CS 598: Spectral Graph Theory. Lecture 3

The Other Eigenvectors and Eigenvalues of the Laplacian

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Today

- Eigenvalue Interlacing
- Fiedler's nodal domain theorem
- Spectra of the Hypercube Graph
- Start on second eigenvalue and importance

Eigenvalue Interlacing

- We will see yet another consequence of Courant-Fischer (proof as exercise in problem set)

Theorem (Eigenvalue Interlacing): Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension n-1 (that is, B is obtained by deleting the same row and column from A). Then

$$\alpha_1 \geq \beta_1$$

Eigenvalue Interlacing

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Theorem (Eigenvalue Interlacing): Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension n-1 (that is, B is obtained by deleting the same row and column from A). Then

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1}$$

Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of A and B resp.

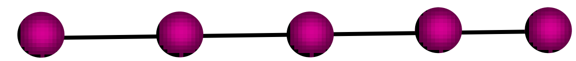
Corollary (Eigenvalue Interlacing): Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension n-k (that is, B is obtained by deleting the same set of k rows and columns from A). Then

$$\alpha_i \geq \beta_i \geq \alpha_{i+k}$$

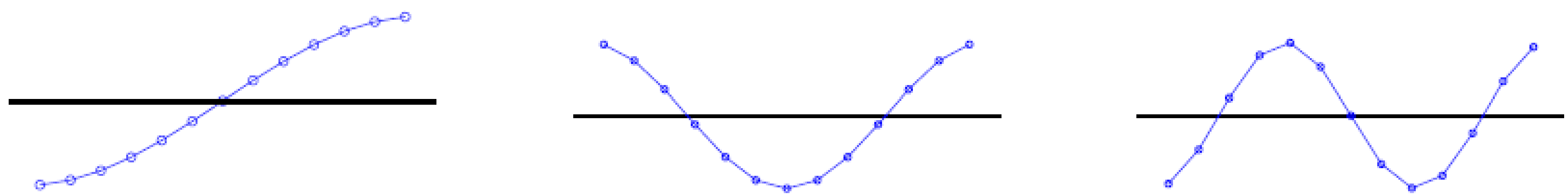
Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of A and B resp.

The Eigenvectors of the Path Graph

Pn:$\{(u, u+1): 0 \leq u < n\}$



- In Lecture 1, we saw: the Laplacian of P_n has eigenvectors $z_k(u) = \sin\left(\frac{\pi k u}{n} + \frac{\pi}{2n}\right)$, for $0 \leq k < n$.



- Here are the first three non-constant eigenvectors of the path graph with 13 vertices. We see that the k-th evector crosses the origin at most k-1 times.

Induced Graph

- Given G=(V,E) and a subset of vertices W a subset of V, the graph induced by G on W is the graph with vertex set W and edge set

$$\{(i,j) \in E, i \in W \ and \ \in j \in W\}$$

The graph is denoted G(W).

Fiedler's Nodal Domain Theorem

- **Theorem.** Let G=(V,E,w) be a weighted connected graph, and let L_G be its Laplacian matrix. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of L_G and $v_1, v_2, \ldots, v_n$ the corresponding eigenvectors. For any k≥2, let $W_k = \{i \in V: v_k(i) \geq 0\}$. Then, the graph induced by G on W_k has at most k-1 connected components.

Proof of Nodal Domain Theorem

We use from previous lecture:

Lemma 1: Perron-Frobenius for Laplacians: Let M be a matrix with non-positive off-diagonal entries s.t. the graph of the non-zero off-diagonal entries is connected. Then the smallest eigenvalue has multiplicity 1 and the corresponding eigenvector is strictly positive

And from this lecture:

Lemma 2: Eigenvalue Interlacing: Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension n-k (that is, B is obtained by deleting the same set of k rows and columns from A). Then $\alpha_i \geq \beta_i \geq \alpha_{i+k}$. Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of A and B resp.

In fact, we will use eigenvalue interlacing when the order of eigenvalues is increasing

Proof of Nodal Domain Theorem

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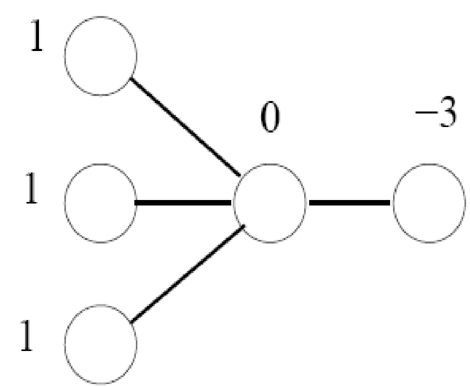
And from this lecture:

Lemma 2. Eigenvalue Interlacing (increasing order version): Let A be an n-by-n symmetric matrix and let B be a principal submatrix of A of dimension n-k (that is, B is obtained by deleting the same set of k rows and columns from A). Then $\alpha_i \le \beta_i \le \alpha_{i+k}$.
Where $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_{n-k}$ are the eigenvalues of A and B resp.

Fiedler's Stronger Nodal Domain Theorem

- **Theorem.** Let G=(V,E,w) be a weighted connected graph, and let L_G be its Laplacian matrix. Let $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of L_G and $v_1, v_2, \ldots, v_n$ the corresponding eigenvectors. For any k≥2, let $W_k = \{i \in V: v_k(i) \geq t\}, t \leq 0$. Then, the graph induced by G on W_k has at most k-1 connected components. (ex)

- The theorem breaks down if we consider $W_k = \{i \in V: v_k(i) > 0\}$, see star graph:



The star graph on 5 vertices, with an eigenvector of $\lambda_2 = 1$.

The Hypercube Graph

- Hypercube H_d is the graph with vertex set $\{0,1\}^d$ and edges between vertices that differ in exactly one bit.

- Alternatively, it is the graph product of the single-edge graph G= ({0,1},{(0,1)})with itself d-1 times, namely:

$$H_d=H_{d-1}xG$$

Graph Products Refresher

- (Definition): Let G(V,E) and H(W,F). The graph product GxH is a graph with vertex set VxW and edge set $((v_1,w),(v_2,w))$ for $(v_1,v_2) \in E$

  $$((v,w_1),(v,w_2)) \text{ for } (w_1,w_2) \in F$$

- If G has evals $\lambda_1,\ldots,\lambda_n$, evecs $x_1,\ldots,x_n$

  H has evals $\mu_1,\ldots,\mu_m$, evecs $y_1,\ldots,y_m$

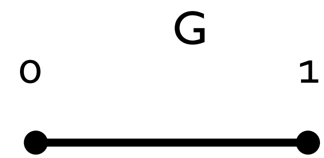
Then GxH has for all i,j in range, an evector

$$z_{ij}(v,w)=x_i(v)y_j(w) \text{ of evalue } \lambda_i + \mu_j$$

- We saw the proof on lecture 1.

The Hypercube Graph

$H_d = H_{d-1} \times G$



- Non-zero eigenvector of the Laplacian of G has eigenvalue 2 (lecture 2)

$L_e =$

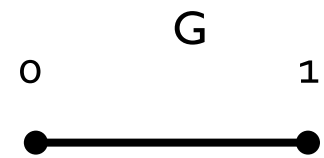
	u	v
u	1	-1
v	-1	1

eigenvalue

eigenvector

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} = 2\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

The Hypercube Graph



$H_d = H_{d-1} x G$

- Non-zero eigenvector of the Laplacian of G has eigenvalue 2 (lecture 2), we see that H_d has eigenvalue 2k with multiplicity $\binom{d}{k}$ for 0≤k ≤d.
- The eigenvectors of H_d are given by the functions

$$v_a(b) = (-1)^{a^T b}$$

Where $a \in \{0,1\}^d$ and we view vertices b as length-d vectors of zeros and ones. The corresponding eigenvalue is for k= number of ones in a. (see blackboard)

The Second Laplacian Eigenvalue and Isoperimerty

- We will now show a basic isoperimetric inequality for the Hypercube graph, using the second eigenvalue.

- Define the boundary of a set of vertices
$$\delta(S) = \{(i,j) \in E : i \in S, j \notin S\}$$

- Theorem: Let G=(V,E) be a graph and let L_G its Laplacian. Let $S \subset V$ and set $\sigma = |S|/|V|$. Then
$$|\delta(S)| \geq \lambda_2 |S|(1-\sigma)$$

- Proof: see blackboard

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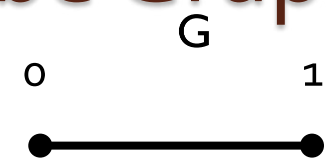
- If second eigenvalue big, then graph well connected.
- Also provides techniques for proving upper bounds on second eigenvalue

Isoperimetry for Hypercube Graph

$H_d = H_{d-1} \times G$



- Non-zero eigenvector of the Laplacian of G has eigenvalue 2 (lecture 2), we see that H_d has eigenvalue 2k with multiplicity $\binom{d}{k}$ for $0 \le k \le d$.

- So λ_2 is 2, which gives from the previous theorem (simple proof of isoperimetic theorem)

$$|\delta(S)| \geq |S|, \text{ for S of size at most } 2^{d-1}.$$

Equality is achieved in dimension cuts

More on λ_2 next lecture (and in fact, the next many lectures!)