CS 598: Spectral Graph Theory. Lecture 10

Expander Graphs and Their Properties

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Today

- Graph approximations. Expanders a approximations of the complete graph
- Quasi-random properties of expanders, expander mixing lemma
- Vertex expansion
- How well can we approximate the complete graph?

Expander Graphs via PRGs

- Our PRG from last lecture uses a random walk on a d-regular G with constant degree.

- We wanted G to be an expander in the following sense: If A_G is G's adjacency matrix and $d = \alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n$ its eigenvalues then we require that $\frac{|\alpha_i|}{d} \leq \frac{1}{10}$.
- This allowed us to show that walk cannot stay in a small set for a long time.

Expander Graphs

- Generally, we define expander graphs to be d-regular graphs whose adjacency matrix eigenvalues satisfy

$$|\alpha_i| \leq \epsilon d$$

for i>1, and some small ϵ.

- We will next see that such graphs are very good approximations of the complete graph.

Graphic Inequalities

- In Lecture 4, we defined

$$G \succcurlyeq H \quad \text{if} \quad L_G \succcurlyeq L_H$$

Or, equivalently, $v^T L_G v \geq v^T L_H v$ for all v

- We say that G is an $\epsilon -$ approximation of H if

$$(1 - \epsilon)H \preccurlyeq G \preccurlyeq (1 + \epsilon)H$$

Approximations of the Complete Graph

- Let G be a d-regular graph whose adjacency eigenvalues satisfy $|\alpha_i| \leq \epsilon d$.
- As its Laplacian eigenvalues satisfy $\lambda_i = d - \alpha_i$, all non-zero eigenvalues are between $(1 - \epsilon)$d and $(1 + \epsilon)d$.
- This means that for all x orthogonal to the all-one's vector

$$(1 - \epsilon)dx^T x \preccurlyeq x^T L_G x \preccurlyeq (1 + \epsilon)dx^T x$$

Approximations of the Complete Graph

- For the complete graph K_n, we know all x orthogonal to the all one's vector satisfy $x^T L_{K_n} x = n x^T x$
- Let H=(d/n)K_n, *so* $x^T L_H x = d x^T x$
- This means that G is an ϵ - approximation of H

$$(1-\epsilon) d x^T x \preccurlyeq x^T L_G x \preccurlyeq (1+\epsilon) d x^T x \Rightarrow$$
$$(1-\epsilon) H \preccurlyeq G \preccurlyeq (1+\epsilon) H \Rightarrow$$
$$-\epsilon H \preccurlyeq G - H \preccurlyeq \epsilon H \Rightarrow \|L_H - L_G\| \leq \epsilon d$$

Quasi-Random Properties of Expander Graphs

- Expanders act like random graphs in many ways.
- We saw last time that with random walk on expander, we can boost the error probability like we could do with random walk on a random graph (or truly random stings, Chernoff bound)
- In fact, a random d-regular graph is expander w.h.p.

Quasi-Random Properties of Expander Graphs

- All sets of vertices in expander graph act like random sets of vertices.
- To see that, consider creating a random set S$\subseteq V$ by including every vertex in S independently w.p. a.
- For every edge (u,v) the probability that each end point is in S is a. Probability that both end points are in S is a^2.
- So, we expect a^2 fraction of the edges to go between vertices in S.
- We show that this is true for all sufficiently large sets in an expander.

Quasi-Random Properties of Expander Graphs: EML

- We show something stronger (expander mixing lemma), for two sets S and T.
- Include each vertex in S w.p. a and each vertex in T w.p. b. We allow vertices to belong to both S and T. We expect that for ab fraction of ordered pairs (u,v) we have u in S and v in T.

Expander Mixing Lemma

- For graph G=(V,E) define the ordered set of pairs
  $\overrightarrow{E(S,T)} = \{(u,v): u \in S, v \in T, (u,v) \in E\}$

- When S, T disjoint $\overrightarrow{|E(S,T)|}$ is the number of edges between S and T.
- $\overrightarrow{|E(S,S)|}$ counts every edge inside S twice.

Expander Mixing Lemma

- **Theorem** (Beigel, Margulis, Spielman'93, Alon, Chung '88)

Let G=(V,E) a d-regular graph that ϵ-approximates the graph $\frac{d}{n}K_n$. Then, for every S⊆V, T⊆V with |S|=an, |T|=bn

$$||\overrightarrow{E(S,T)}| - d\frac{|S||T|}{n}| \leq \epsilon d\sqrt{|S||T|} \Rightarrow$$

$$||\overrightarrow{E(S,T)}| - dabn| \leq \epsilon dn\sqrt{ab}$$

Expander Mixing Lemma(stronger version)

- **Theorem** (Beigel, Margulis, Spielman'93, Alon, Chung '88)

Let G=(V,E) a d-regular graph that ϵ-approximates the graph $\frac{d}{n}K_n$. Then, for every S⊆V, T⊆V with |S|=an, |T|=bn

$$||\overrightarrow{E(S,T)}| - dabn| \leq \epsilon dn\sqrt{(a-a^2)(b-b^2)}$$

Vertex Expansion

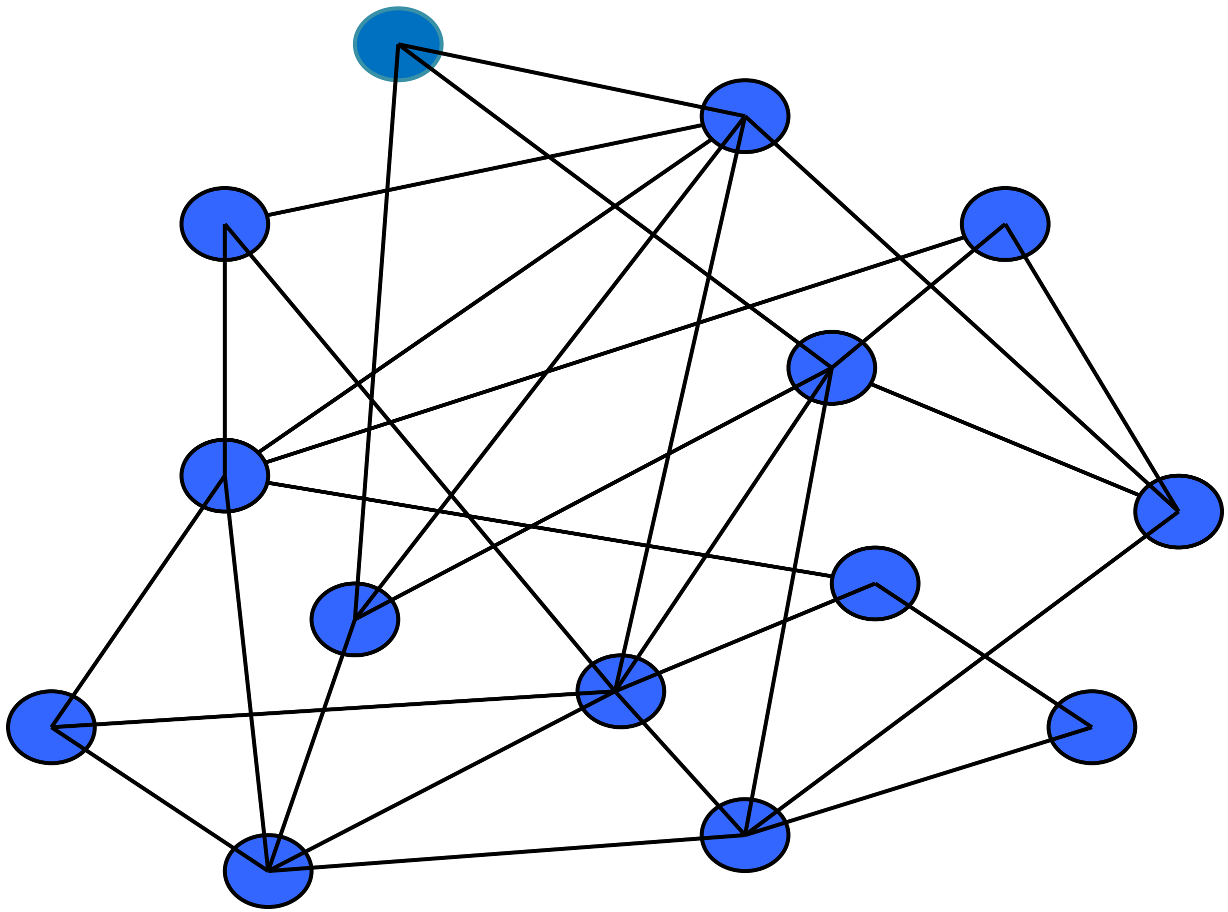
- The name Expander came from the fact that small sets of vertices have unusually large numbers of neighbors. For a subset S of vertices, let N(S) denote the set of vertices that are neighbors of vertices in S.
- **Theorem** (Tanner '84).

Let G=(V,E) a d-regular graph that ϵ-approximates the graph $\frac{d}{n}K_n$. Then, for every S⊆V, $|S|=an$,

$$|N(S)| \geq \frac{|S|}{\epsilon^2(1-a)+a}$$

How Well Can a Graph Approximate the Complete Graph?

G



How Well Can a Graph Approximate the Complete Graph?

- Apply Tanner for S={v}, a single vertex.
- Since v has exactly d neighbors, we find that $\epsilon^2\left(1-\frac{1}{n}\right)+\frac{1}{n} \geq \frac{1}{d} \Longrightarrow \epsilon \gtrsim \frac{1}{\sqrt{d}}$
- But how small can it get?
- Ramanujan graphs [Margulis, LPS '88] give $\Longrightarrow$ $\epsilon \leq \frac{2\sqrt{d-1}}{d}$
- We will see that if d fixes as n grows, the bound is tight in the limit.