CS 598: Spectral Graph Theory. Lecture 1

The Laplacian

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Administrativia

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- office hours: by appointment
- course website: https://courses.engr.illinois.edu/cs598sgt
- scribe notes (?), homeworks (2-3), projects (instead of final exam), prerequisites.
- Course goals.

Course Overview (rough, could be modified)

- Graphs, matrices and their spectra (3-4 Lectures)
- Topics on the second eigenvalue and expanders (10-12 Lectures)
- Topics on random graphs and their spectrum (3-4 Lectures)
- Topics on all eigenvalues-graph approximations (3-4 Lectures)

Block 1: Graphs, Matrices and their Spectra

- Adjacency matrix, diffusion operator, Laplacian.
- Eigenvalues and eigenvectors of graphs, examples.
- Properties of the Laplacian, properties of adjacency matrix and their relations.
- Courant-Fischer, Perron-Frobenius, nodal domains, interlacing.
- Eigenvalue bounding techniques, examples.

Block 2: Second Eigenvalue, Expanders

- Edge expansion, graph cutting, Cheeger's inequality.
- Semidefinite programming, duality and connections with the second eigenvalue.
- Random Walks and Convergence.
- Expanders: existence, constructions and applications, graph lifts.
- Ramanujan expanders: existence (LPS, MSS)

Block 3: Random Graph Spectra

- Random graphs are expanding.
- Trace Method.
- **ε-**nets.
- Matrix Bernstein.
- Random regular graphs.
- Random Lifts (maybe).

Block 4: Graph Approximations

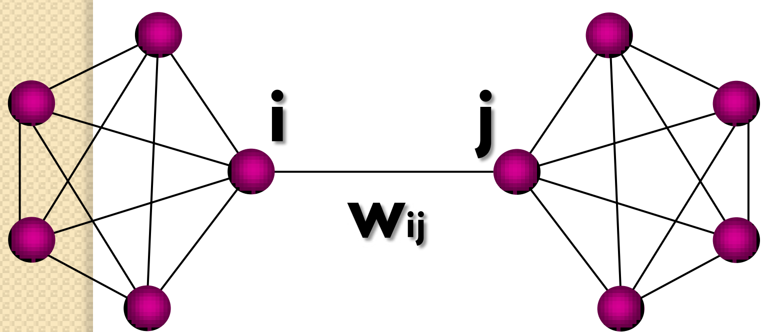
- Various graph approximations, sparsification and applications.
- Sparsification via Alshwade-Winter.
- Resistance Distance.
- Spectral sparsifiers with effective resistances.
- Deterministic algorithm for spectral sparsification (BSS)

In the next few minutes:

Why spectral graph theory is both natural and magical

Representing Graphs

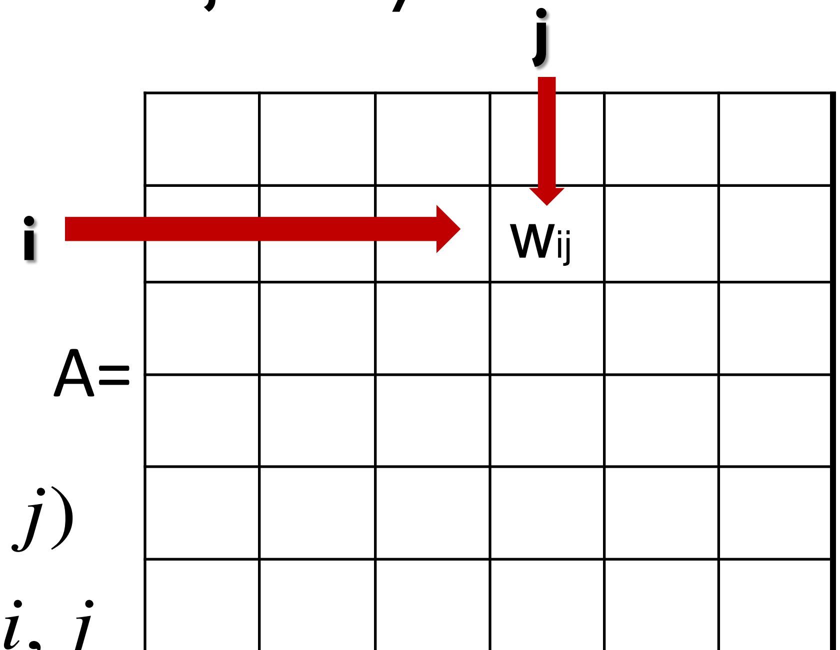
Obviously, we can represent a graph with an nxn matrix



V: n nodes
E: m edges

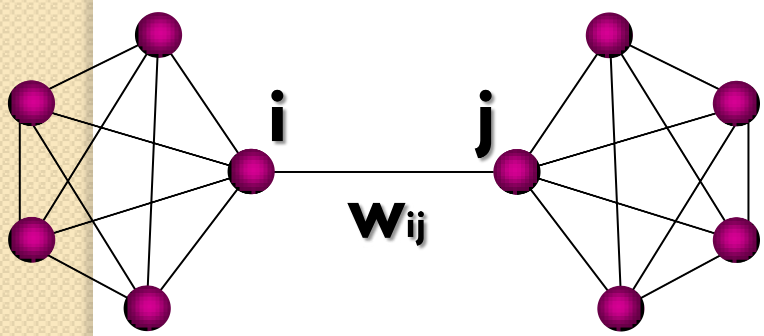
G = {V,E}

Adjacency matrix



$$A_{ij} = \begin{cases} w_{ij} & \text{weight of edge } (i, j) \\ 0 & \text{if no edge between } i, j \end{cases}$$

Representing Graphs



V: n nodes
E: m edges

G = {V,E}

Obviously, we can represent a graph with an nxn matrix

What is not so obvious, is that once we have matrix representation view graph as **linear operator**

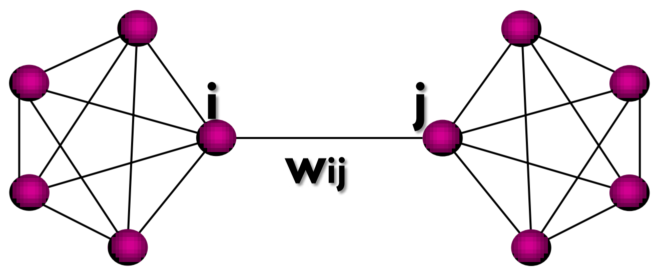
- Can be used to multiply vectors.
- Vectors that don't rotate but just scale = eigenvectors.
- Scaling factor= eigenvalue

$$A : \Re^n \rightarrow \Re^n$$

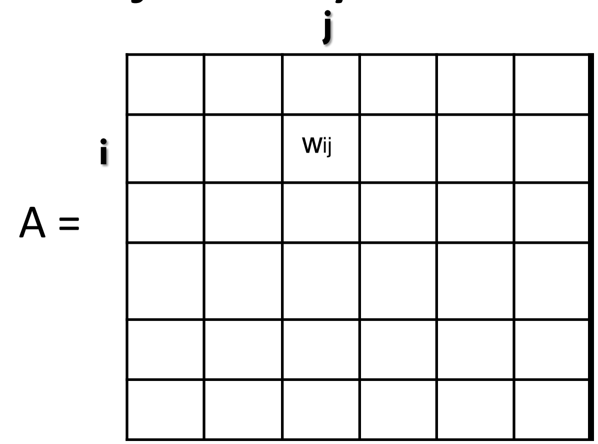
$$Ax = \mu x$$

Amazing how this point of view gives information about graph

"Listen" to the Graph



Adjacency matrix



List of eigenvalues

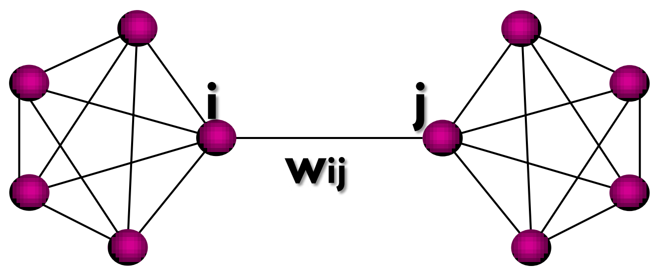
$\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}$:graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

Hear shape of the drum

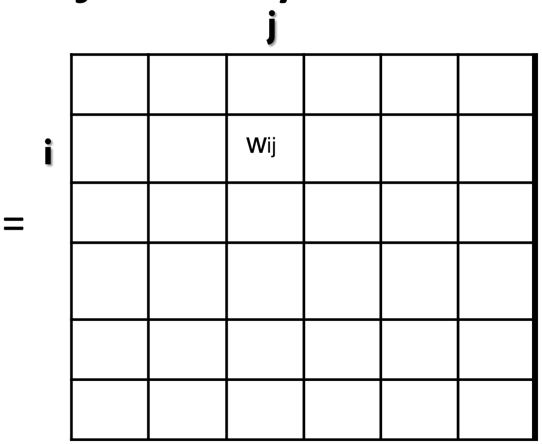
A drum:

"Listen" to the Graph



Adjacency matrix

A =



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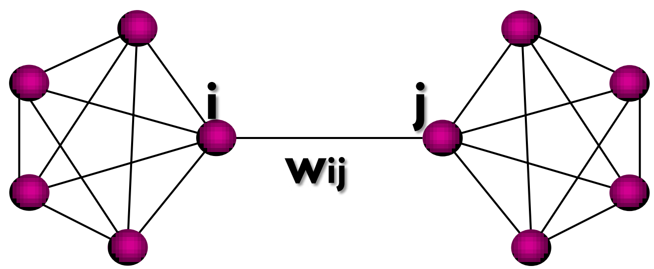
Hear shape of the drum

Its sound:

"Listen" to the Graph



Adjacency matrix

A =

		j			
i		Wij			

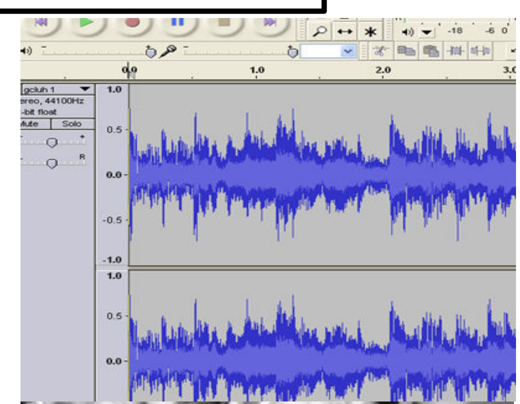
List of eigenvalues

$\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}$:graph SPECTRUM

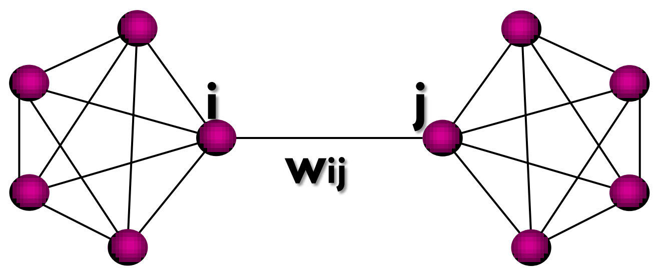
Eigenvalues reveal global graph properties not apparent from edge structure

Hear shape of the drum

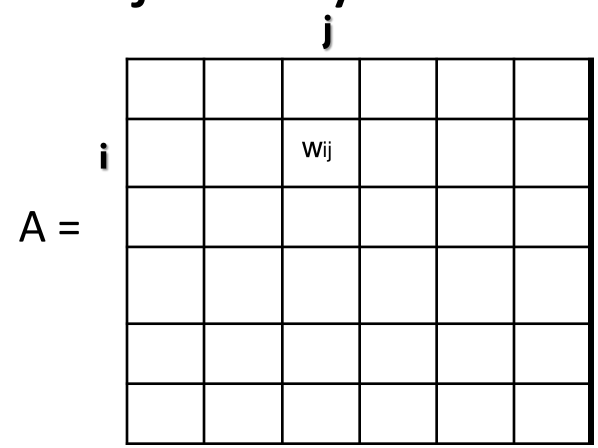
Its sound (eigenfrequenies):



“Listen” to the Graph



Adjacency matrix

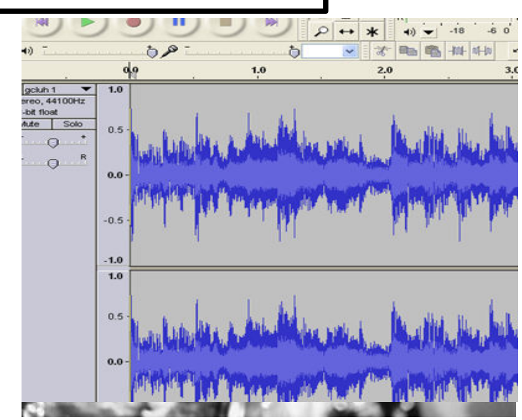


List of eigenvalues

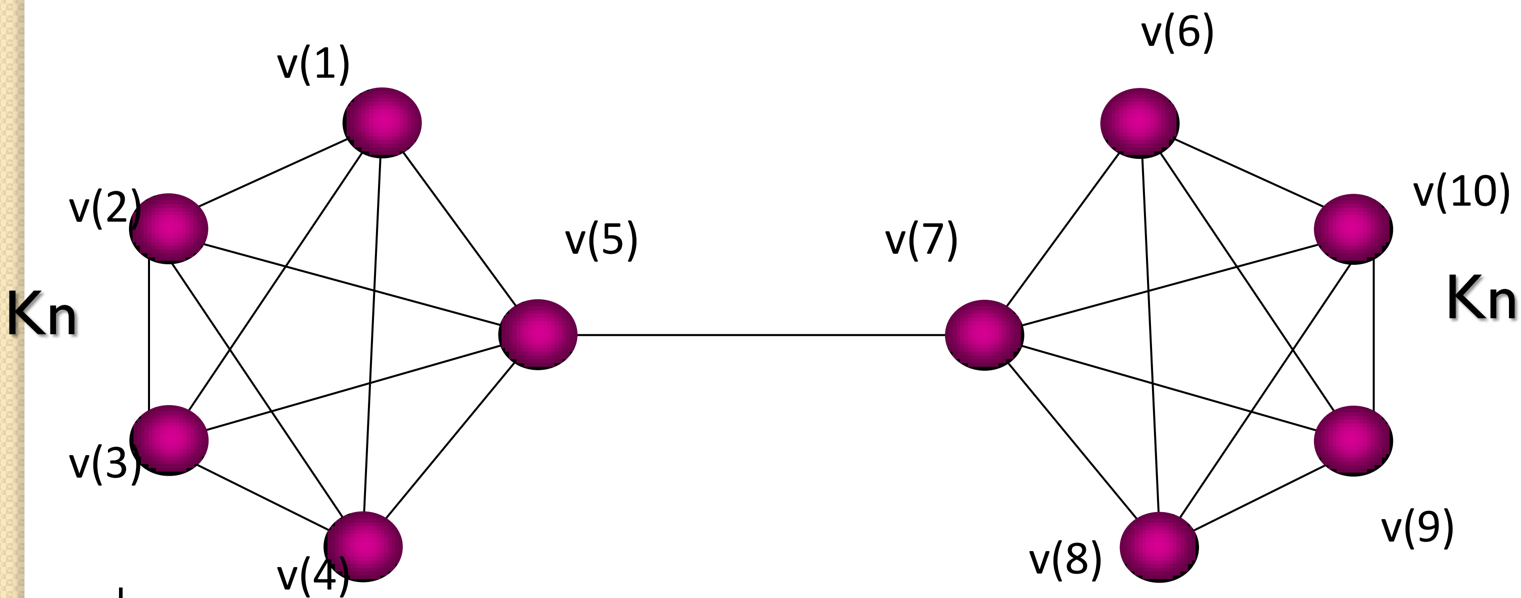
$\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}$:graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

If graph was a drum, spectrum would be its sound



Eigenvectors are Functions on Graph



V: 2n nodes

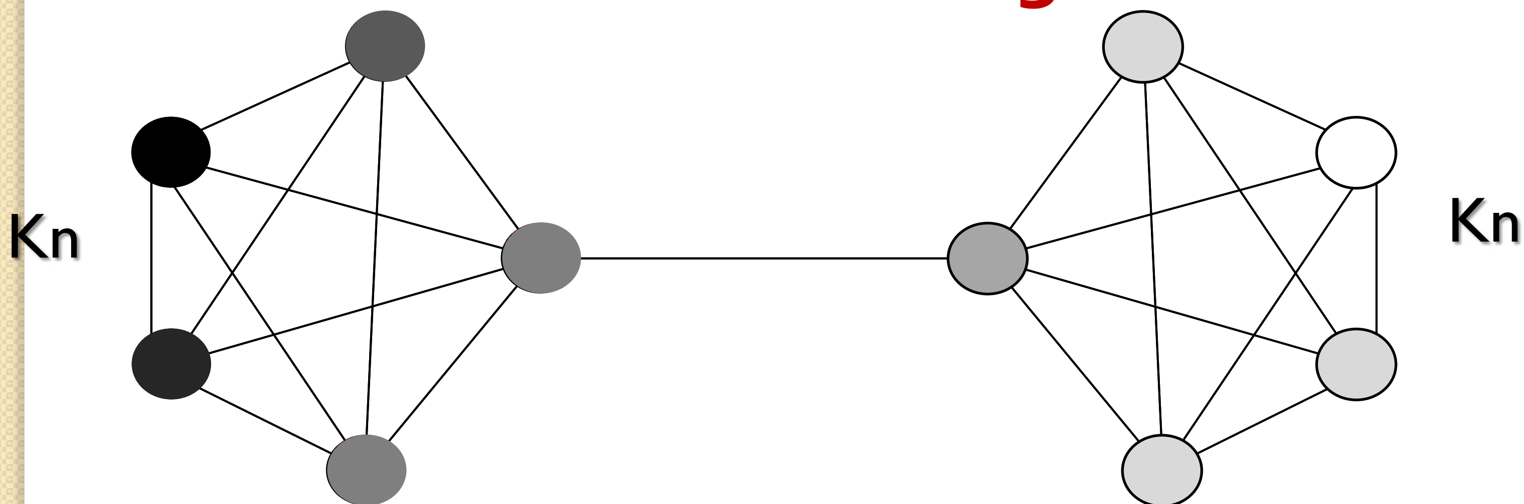
$$Av = \mu v$$

$$v \in \Re^n, \quad v : V \to \Re$$

$v(i) =$ value at node i

Eigenvectors are Functions on Graph

"Coloring"



V: 2n nodes

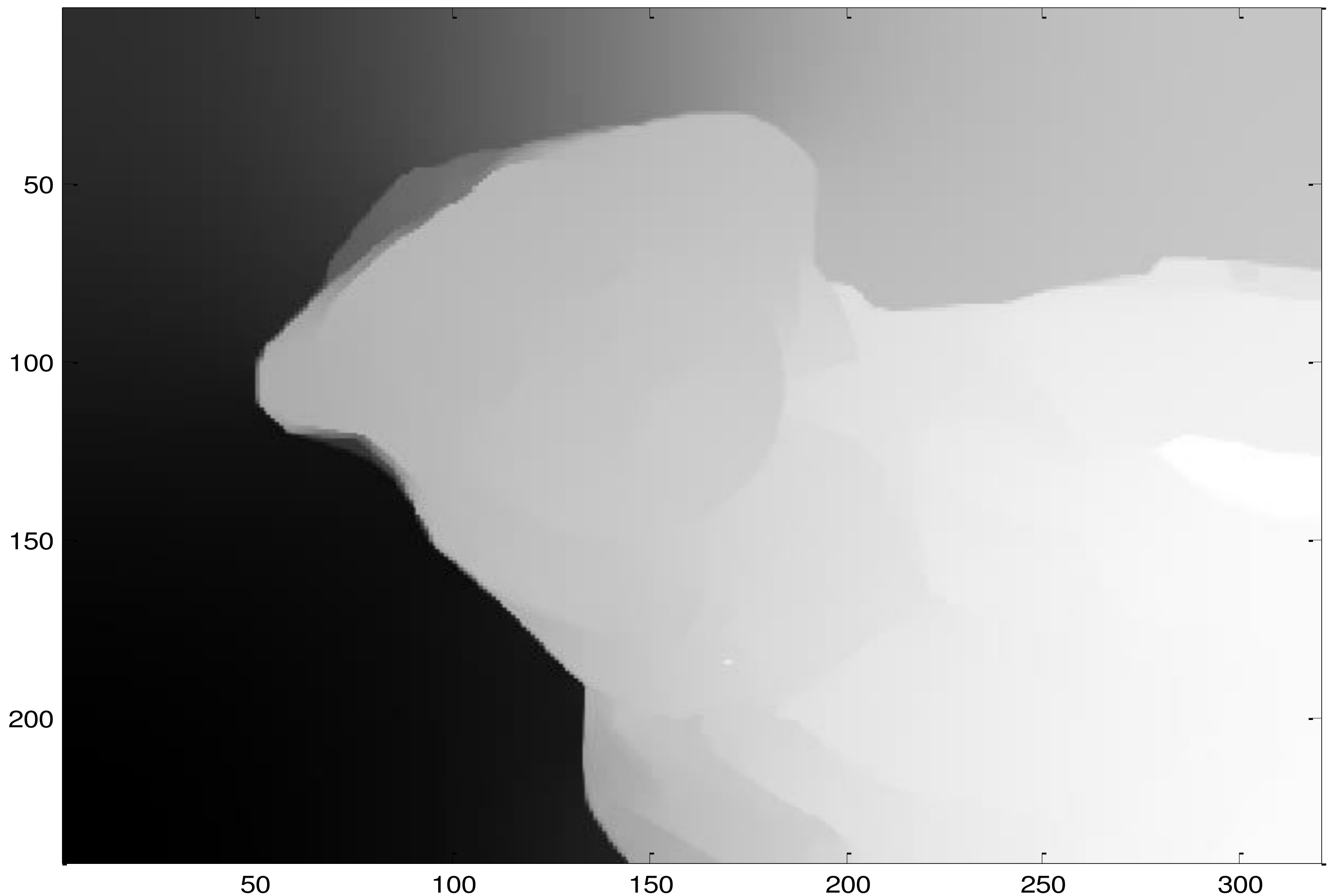
$$v \in \Re^n, \quad v : V \to \Re \qquad Av = \mu v$$

$v(i) =$ value at node i = different shade of grey

So, let's See the Eigenvectors

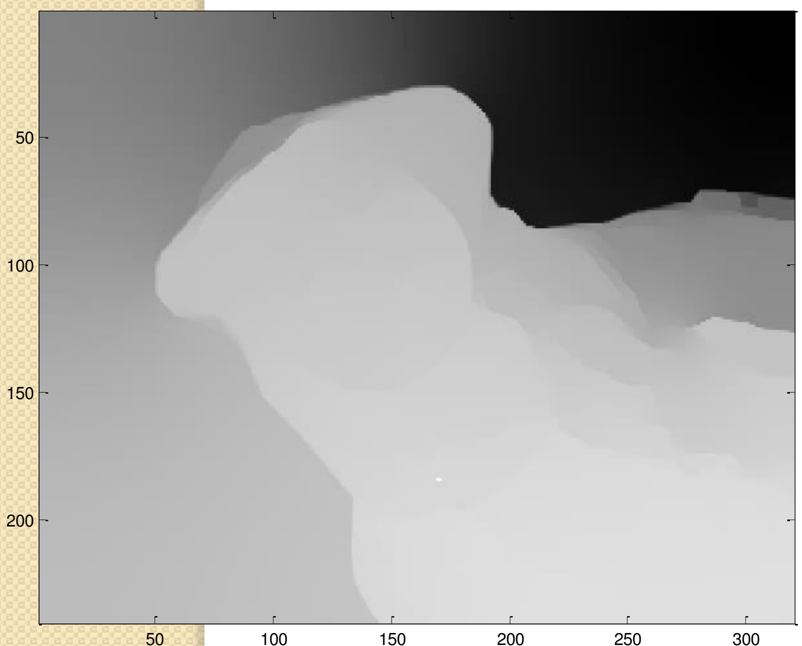
* Slides from Dan Spielman

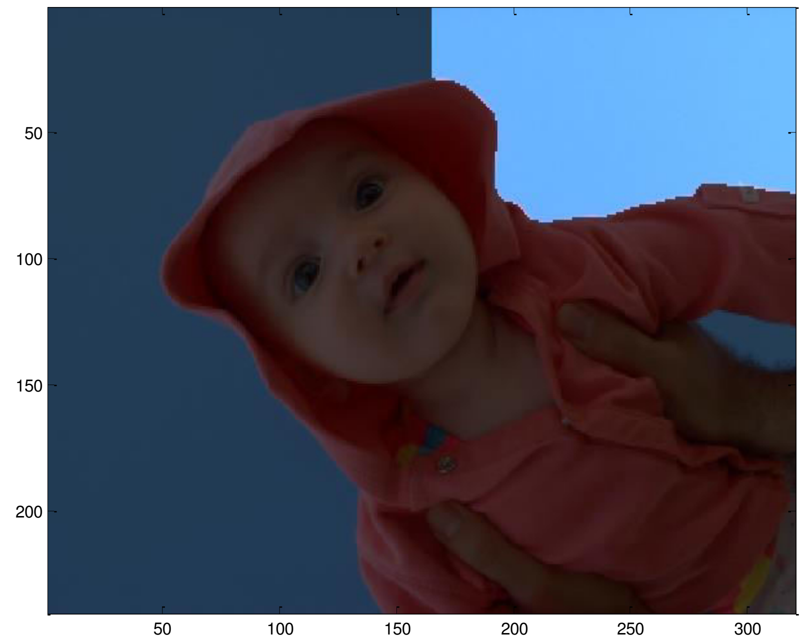
The second eigenvector



* Slides from Dan Spielman

Third Eigenvector





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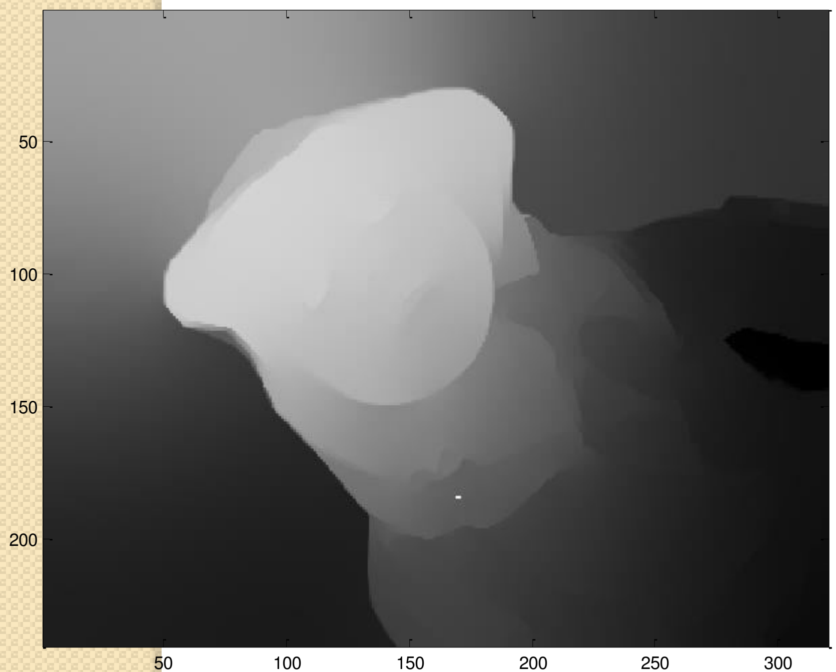
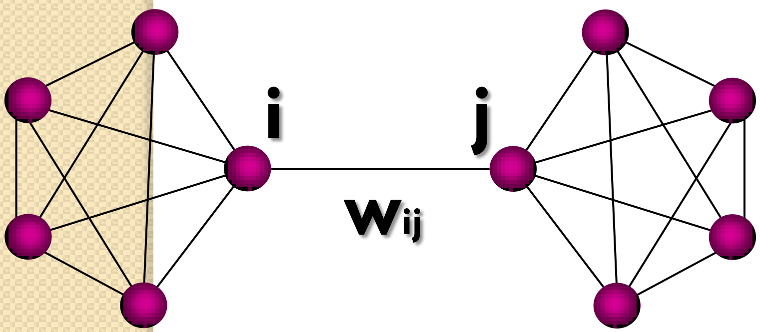
Fourth Eigenvector



* Slides from Dan Spielman

Another view: the Laplacian



G = {V,E}

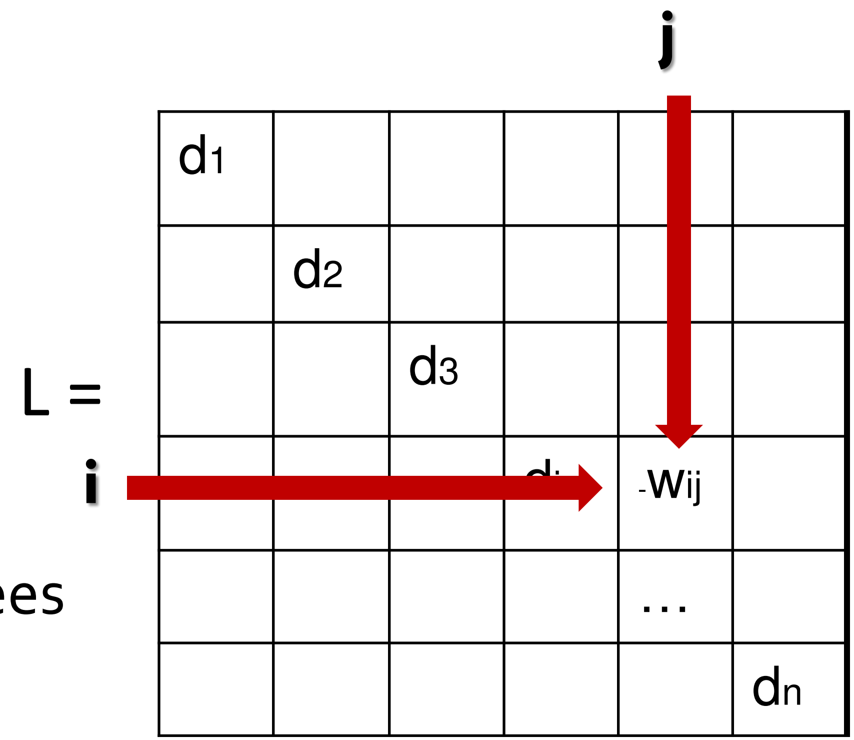
We can also view graph as Laplacian

$$L = D - A$$

where D is diagonal matrix of degrees

$$L_{ij} = \begin{cases} d_i & if\ \ i = j \\ -w_{ij} & if\ \ (i,j)\ edge \\ 0 & otherwise \end{cases}$$

Laplacian



The Laplacian: Fast Facts

$$L\mathbf{1} = \mathbf{0}$$

so, zero is an eigenvalue

1 an eigenvector

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

SPECTRUM of the Laplacian

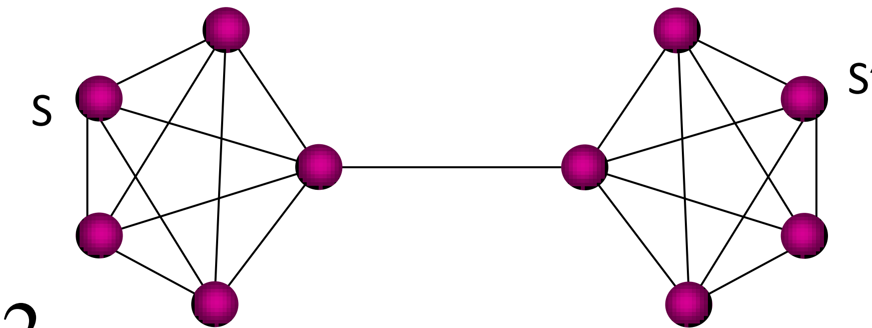
$\lambda_2 > 0$ ⟺ Graph CONNECTED

λ_2 also "algebraic connectivity"

The further from **0**, the more connected

Cuts and Algebraic Connectivity

Cuts in a graph:



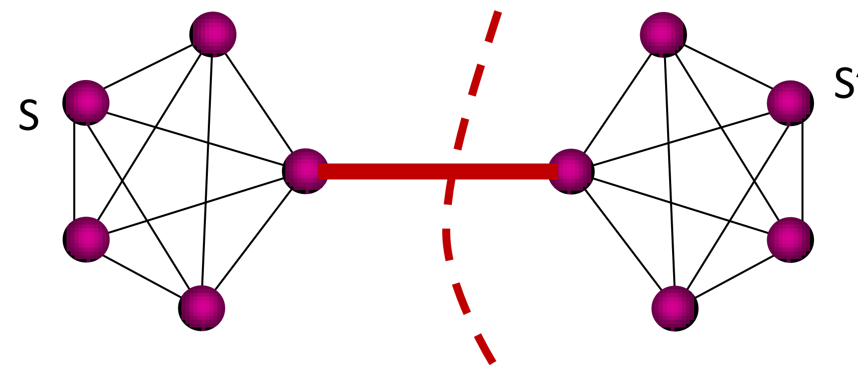
$$cut(S,S') = \frac{E(S,S')}{|S|}, |S| \quad n/2$$

Graph not well-connected when “easily” cut in two pieces

Cuts and Algebraic Connectivity

Sparsest Cut:

$$h(G) = \min_{S:|S| \le n/2} \frac{E(S, \bar{S})}{|S|}$$



Graph not well-connected when "easily" cut in two pieces

Would like to know Sparsest Cut but NP hard to find

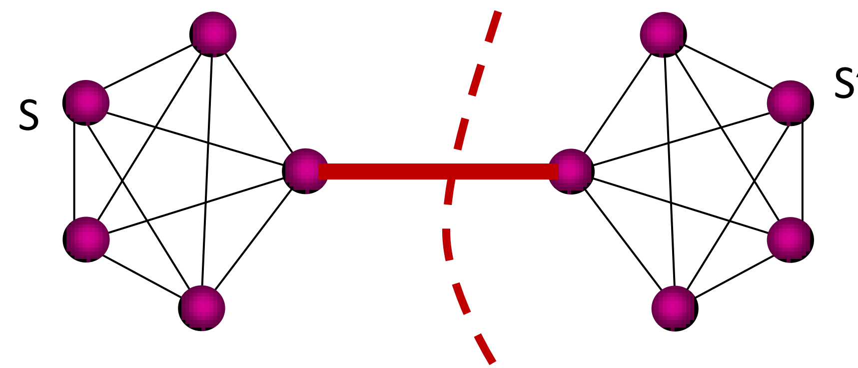
How does algebraic connectivity relate to standard connectivity?

Theorem(Cheeger-Alon-Milman): $\lambda_2 \le h(G) \le \sqrt{2d_{\max}}\sqrt{\lambda_2}$

Cuts and Algebraic Connectivity

Sparsest Cut:

$$h(G) = \min_{S:|S| \le n/2} \frac{E(S, \bar{S})}{|S|}$$



Graph not well-connected when “easily” cut in two pieces

Would like to know Sparsest Cut but NP hard to find

How does algebraic connectivity relate to standard connectivity?

Algebraic connectivity large ⟷ Graph well-connected

Today

- More on evectors and evalues
- The Laplacian, revisited
- Properties of Laplacian spectra, PSD matrices.
- Spectra of common graphs.
- Start bounding Laplacian evalues

Evectors and Evalues

- Vector v is evector of matrix A with evalue μ if $Av=\mu v$.
- We are interested (almost always) in symmetric matrices, for which the following special properties hold:
  - If v_1, v_2 are evectors of A with evalues μ_1, μ_2 and $\mu_1 \neq \mu_2$, then v_1 is orthogonal to v_2. (Proof)

  - If v_1, v_2 are evectors of A with the same evalue μ, then v_1+v_2 is as well. The multiplicity of evalue μ is the dimension of the space of evectors with evalue μ.

Evectors and Evalues

- Vector v is evector of matrix A with evalue μ if Av=μv.
- We are interested (almost always) in symmetric matrices, for which the following special properties hold:

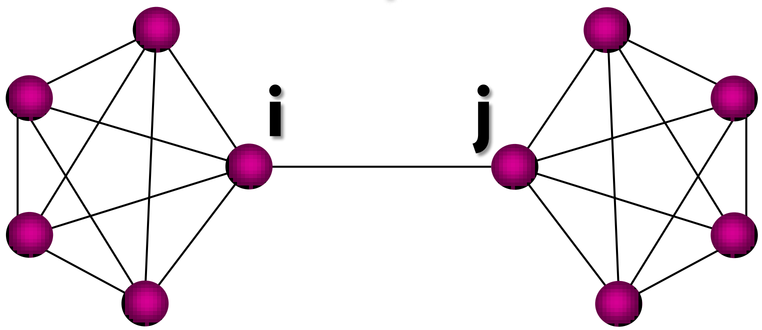
  - Every n-by-n symmetric matrix has n evalues $\{\mu_1 \leq \cdots \leq \mu_n\}$ counting multiplicities, and and orthonormal basis of corresponding evectors $\{v_1 \leq \cdots \leq v_n\}$, so that

    $Av_i = \mu_i v_i$

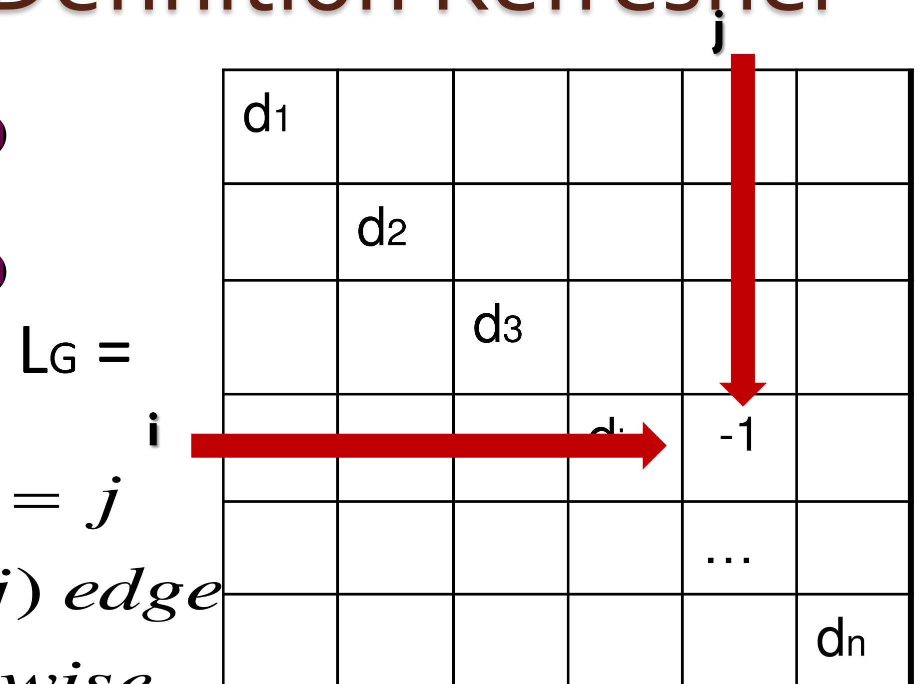
  - If we let V be the matrix whose i-th column is v_i , and M the diagonal matrix whose i-th diagonal is μ_i, we can compactly write AV=VM. Multiplying by on the right, we obtain the eigendecomposition of A:

$$A = AVV^T = VMV^T = \sum_i \mu_i v_i v_i^T$$

The Laplacian: Definition Refresher



G = {V,E}



$$L_G(i,j) = \begin{cases} d_i & if \;\; i = j \\ -1 & if \;\; (i,j) \; edge \\ 0 & otherwise \end{cases}$$

Where d_i is the degree of i-th vertex.
For convenience, we have unweighted graphs

- D_G = Diagonal matrix of degrees
- A_G = Adjacency matrix of the graph
- $L_G = D_G - A_G$

Redefining the Laplacian

- Let L_e be the Laplacian of the graph on n vertices consisting of just one edge e=(u,v).

$$L_e(i,j) = \begin{cases} 1 & \text{if } i = j, i \in u, v \\ -1 & \text{if } i = u, j = v, \text{or vice versa} \\ 0 & \text{otherwise} \end{cases}$$

$$L_e = \begin{array}{c|cc} & u & v \\ \hline u & 1 & -1 \\ v & -1 & 1 \end{array} \oplus [zeros]$$

- For a graph G with edge set E we now define

$$L_G = \sum_{e \in E} L_e$$

- Many elementary properties of the Laplacian now follow from this definition as we will see next (prove facts for one edge and then add).

Laplacian of an edge, contd.

L_e =

	u	v
u	1	-1
v	-1	1

$\otimes [zeros]$

eigenvalue

eigenvector

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = 2 \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

- Since evalues are zero and 2, we see that L_e is P.S.D. Moreover,

$$x^T L_e x = (x_1 x_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 - x_2)^2$$

Review of Positive Semidefiniteness

- **Definition:** A symmetric matrix M is positive semidefinite (PSD) if:

$$x^T M x \geq 0 \ \forall x \in R^n$$

Positive definite (PD) if inequality is strict for all $x \neq 0$.

- PSD iff all evalues are non-negative (exercise.)

- PSD iff M can be written as $M = A^T A$, where A can be n-by-k (not necessarily symmetric) and is not unique.

  Proof: see blackboard

More Properties of Laplacian

From the definition using edge sums, we get:

- **(PSD-ness)**The Laplacian of any graph is PSD.

$$x^T L_G x = x^T \left(\sum_{\{e \in E\}} L_e \right) x = \sum_{e \in \mathrm{E}} x^T L_e x$$

$$= \sum_{i,j \in E} \left(x_i - x_j \right)^2$$

- **(Connectivity)** G is connected iff **λ2** positive or alternatively, the null space of the Laplacian of G is 1-dimensional and spanned by the vector **1.** (Proof on blackboard)

- **Corollary:** The multiplicity of zero as an eigenvalue equals the number of connected components of the graph.

More Properties of Laplacian

- **(Edge union)** If G and H are two graphs on the same vertex set, with disjoint edge set then

$$L_{G \cup H} = L_G + L_H \ \text{(additivity)}$$

- If a vertex is isolated, the corresponding row and column of Laplacian are zero

- **(Disjoint union)** Together these imply that for the disjoint union of graphs G and H

$$L_{G \coprod H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$$

More Properties of Laplacian

- **(Edge union)** If G and H are two graphs on the same vertex set, with disjoint edge set then

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- **(Disjoint union)** Together these imply that for the disjoint union of graphs G and H

$$L_{G \coprod H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix}$$

- **(Disjoint union spectrum)** If L_G has evectors $v_1, \ldots, v_n$ with evalues $\lambda_1, \ldots, \lambda_n$ and L_H has evectors $w_1, \ldots, w_n$ with evalues $\mu_1, \ldots, \mu_n$ then $L_G \coprod L_H$ has evectors

$v_1 \oplus 0, \ldots, v_n \oplus 0, 0 \oplus w_1, \ldots, 0 \oplus w_n$ with evalues $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n.$

The Incidence Matrix: Factoring the Laplacian

- We can factor L as $L = V^T \Lambda V$ using evectors but also exists nicer factorization
- Define the incidence matrix B to be the m-by-n matrix

$$B(e,v) = \begin{cases} 1, if\ e = (v,w)\ and\ v < w \\ -1, if\ e = (v,w)\ and\ v > w \\ 0\ o.w. \end{cases}$$

- Example of incidence matrix

$$\mathrm{B} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \text{ and } \mathrm{L} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

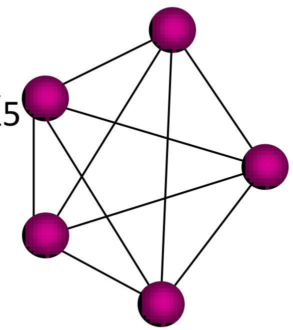
- Claim: $L = B^T B$(exercise)
- Gives another proof that L is PSD.

Spectra of Some Common Graphs

- The complete graph K_n on n vertices with edge set $\{(u, v): u \neq v\}$
- The path graph P_n on n vertices with edge set $\{(u, u+1): 0 \leq u < n\}$
- The ring graph R_n on n vertices with edge set $\{(u, u+1): 0 \leq u < n\} \cup (0, n-1)$
- The grid graph G_{nxm} on nxm vertices with edges from each node (x,y) to nodes that differ by one in just one coordinate
- Product graphs in general

The Complete Graph

$K_n: \{(u, v): u \neq v\}$

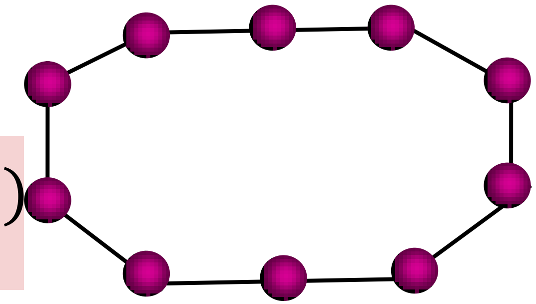


- The Laplacian of K_n has eigenvalue zero with multiplicity 1 (since it is connected) and n with multiplicity n-1.

- Proof: see blackboard

The Ring Graph

R10



Rn: $\{(u, u+1): 0 \leq u < n\} \cup (0, n-1)$

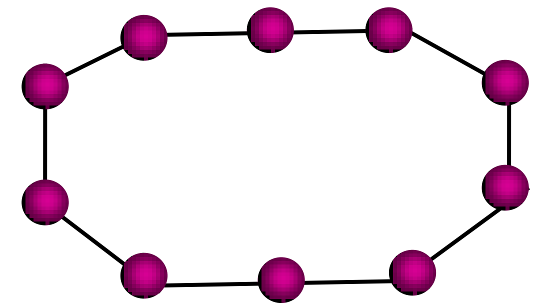
- The Laplacian of Rn has eigenvectors

$$x_k(u) = \sin\left(\frac{2\pi k u}{n}\right) \; and \; y_k(u) = \cos\left(\frac{2\pi k u}{n}\right)$$

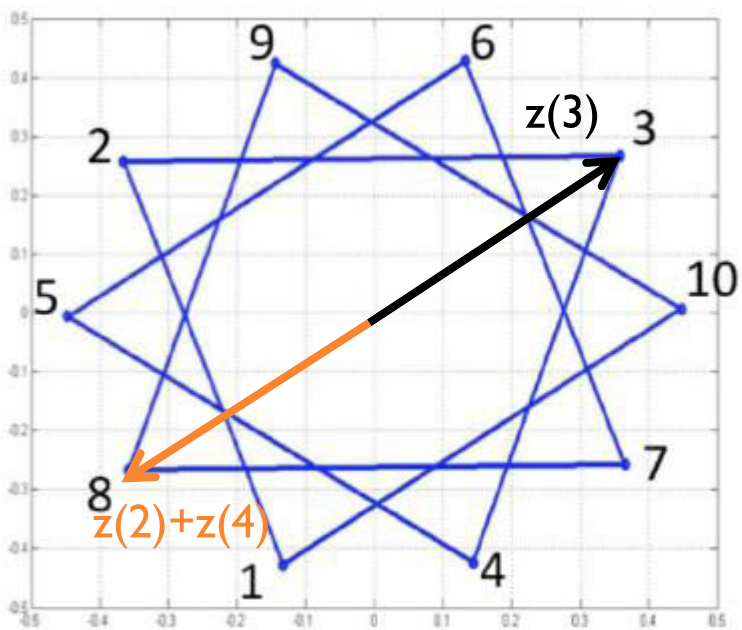
for $k \leq n/2$. Both have eigenvalue $2 - 2\cos\left(\frac{2\pi k}{n}\right)$. Note x_0 should be ignored and y_0 is the all ones vector. If n is even, then $x_{n/2}$ should be ignored.

Proof: By plotting the graph on the circle using these vectors as coordinates.

The Ring Graph



Spectral embedding for k=3



Let z(u) be the point (x_k(u), y_k(u)) on the plane.

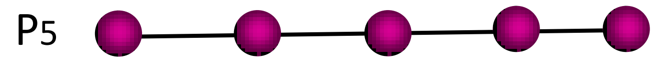
Consider the vector z(u-1) - 2 z(u) + z(u+1). By the reflection symmetry of the picture, it is parallel to z(u)

Let z(u-1) - 2 z(u) + z(u+1) = λz(u). By rotational symmetry, the constant λ is independent of u.

To compute λ consider the vertex u=1.

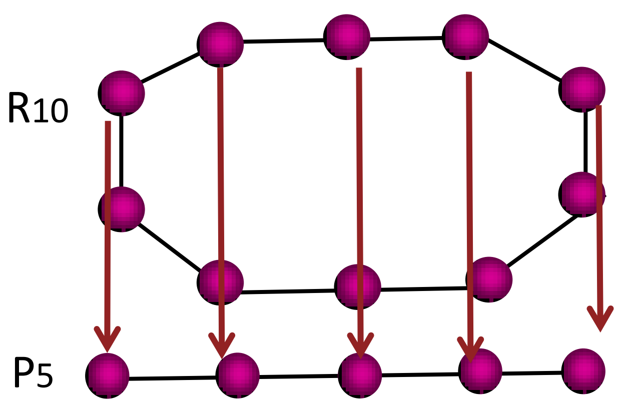
The Path Graph

Pn: $\{(u, u+1): 0 \leq u < n\}$



- The Laplacian of P_n has the same eigenvalues as R_{2n} and eigenvectors $z_k(u) = \cos\left(\frac{\pi k u}{n} - \frac{\pi k}{2n}\right)$, for k<n.

Proof: Treat P_n as a quotient of R_{2n}. Use projection



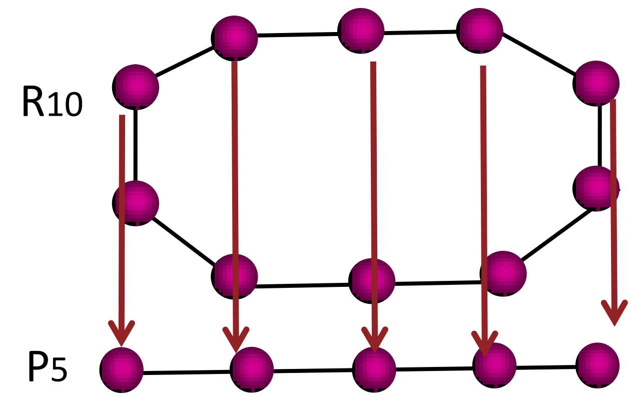
$$f: R_{2n} \to P_n$$

$$f(u) = \begin{cases} u, if\ u < n \\ 2n + 1 - u, if\ u \geq n \end{cases}$$

The Path Graph

Proof: Treat P_n as a quotient of R_{2n}.

Use projection



- Let z be an eigenvector of the ring, with z(u)=z(2n+1-u) for all u.
- Take the first n components of z and call this vector v.
- To see that v is an eigenvector of P_n, verify that it satisfies for some λ:

$$2v(u)-v(u-1)-v(u+1)= \lambda v(u), \text{ for } 0<u<n-1$$
$$v(0)-v(1)= \lambda v(1)$$
$$v(n-1)-v(n-2)= \lambda v(n-1)$$

- Take z as claimed, i.e. which is in the span of x_k and y_k.
- (verify details as exercise)

Graph Products

- (Definition): Let G(V,E) and H(W,F). The graph product GxH is a graph with vertex set VxW and edge set $((v_1,w),(v_2,w))$ for (v_1,v_2)

  $$((v,w_1),(v,w_2)) \text{ for } (w_1,w_2)$$

- If G has evals $\lambda_1,\ldots,\lambda_n$, evecs $x_1,\ldots,x_n$

  H has evals $\mu_1,\ldots,\mu_m$, evecs $y_1,\ldots,y_m$

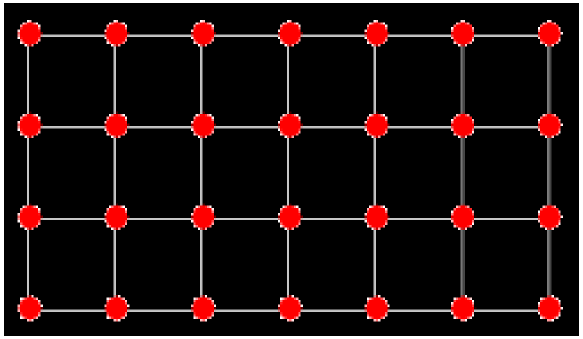
Then GxH has for all i,j in range, an evector

$$z_{ij}(v,w)=x_i(v)y_j(w) \text{ of evalue } \lambda_i + \mu_j$$

- Proof: see blackboard

Graph Products: Grid Graph

$$G_{n\times m} = P_n \times P_m$$



- Immediately get spectra from path.

Start Bounding Laplacian Eigenvalues

Sum of Eigenvalues, Extremal Eigenvalues

- $\sum_i \lambda_i = \sum_i d_i \leq d_{max} n$ where di is the degree of vertex i.

  Proof: take the trace of L

- $\lambda_2 \leq \frac{\sum_i d_i}{n-1}$ and $\lambda_n \geq \frac{\sum_i d_i}{n-1}$

  Proof: previous inequality + $\lambda_1 = 0$.

Courant-Fischer

- For any nxn symmetric matrix A with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \ldots, v_n$, denote S_k the span of $v_1, v_2, \ldots, v_k$ and ${S_k}^{\perp}$ the orthogonal complement , then

$$\alpha_k = \max_{x \in S_{k-1}^{\perp}, x \neq 0} \frac{x^T A x}{x^T x} \qquad \alpha_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

Proof: see blackboard

Courant-Fischer

- **Courant-Fischer Min Max Formula:** For any nxn symmetric matrix A with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \ldots, v_n$, denote S_k the span of $v_1, v_2, \ldots, v_k$ and ${S_k}^{\perp}$ the orthogonal complement , then

$$\alpha_k = \max_{S \subseteq R^n, \dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_k = \min_{S \subseteq R^n, \dim(S)=n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}$$

Proof: see blackboard

Courant-Fischer for Laplacian

- Courant-Fischer Min Max Formula for increasing evalue order (e.g. Laplacians): For any nxn symmetric matrix L, with eigenvalues in increasing order

$$\lambda_k = \min_{S \text{ of } \dim k} \max_{x \in S} \frac{x^T L x}{x^T x}$$

$$\lambda_k = \max_{S \text{ of } \dim n-k-1} \min_{x \in S} \frac{x^T L x}{x^T x}$$

- Definition (Rayleigh Quotient): The ratio $\frac{x^T L x}{x^T x}$ is called the *Rayleigh Quotient* of x with respect to L.
- Next lecture we will use it to bound evalues of Laplacians of certain graphs.