CS 598. Spectral Graph Theory
Problem Set 2

Alexandra Kolla

due March 16, 2015

Problem 1 *Alon Boppana*

(5 pts.) Show that there exists a constant c, such that for every finite connected d-regular graph G with diameter D, the second eigenvalue of the adjacency matrix of G satisfies the following bound:

$$\lambda_2(G) \geq 2\sqrt{d-1}(1 - \frac{c}{D^2})$$

Please try to do this without looking at the Alon Boppana proof, following the style of proof that we talked about in Lecture 10.

Problem 2 *Abelian Cayley Graphs*

(5 pts.)

Let G be an Abelian group on n elements, and H a Cayley graph of G of degree d with second adjacency matrix eigenvalue less than $(1-\epsilon)d$ for some constant ϵ. Show that $d \geq \Omega_\epsilon(\log n)$. Namely, in order to construct expanders from abelian Cayley graphs, the degree needs to be logarithmic.

Problem 3 *Diameter and Multiplicity*

(7 pts.) Let G be any graph, and let $L = D - A$ be the Laplacian, where D is the diagonal degree matrix and A is the adjacency matrix. You will prove that, for any $r \geq 1$, if L has at most $r+1$ distinct eigenvalues, then the diameter of G is at most r. In particular, this implies that the only graph with exactly two distinct eigenvalues is the complete graph. Let D be the diameter of G.

1. Prove that if p is a polynomial of degree m such that $p(A)$ has all non-zero entries, then $D \leq m$.

2. Prove the same thing for L instead of A.

3. Now, let $0 = \lambda_1 \leq \lambda_2, \cdots, \leq \lambda_n$ be the eigenvalues of L with corresponding orthogonal eigenvectors $u_1, u_2, \cdots, u_n$. Prove that for any polynomial p such that $p(0) = 1$, we have

$$p(L) = \frac{J}{n} + \sum_{i=1}^{n} p(\lambda_i) u_i u_i^T$$

where J is the all one's matrix.

4. Using the fact that L has only $r+1$ distinct eigenvalues, use the preceding two parts to show that $D \leq r$.

Problem 4 *Generalized Cheeger Rounding*

(8 pts.)

Let G be an n-node, undirected, connected graph (you can also assume it is unweighted if you want) with adjacency matrix A. Define the normalized K-cut of a graph, into $K \geq 2$ disjoint sets of vertices $S_1, S_2, \cdots, S_K$, to be

$$NCUT(S_1, S_2, \cdots, S_K) = \sum_{i=1}^{K} \frac{Cut(S_i)}{Vol(S_i)}$$

where $Cut(S) = \sum_{i \in S, j \in V \setminus S} A(i,j)$ and $Vol(S) = \sum_{i \in S} d_i$. Here d_i is the degree of vertex i. Let h_i be an "indicator vector" for set S_i as follows:

$$h_i(v) = \begin{cases} \frac{1}{\sqrt{Vol(S_i)}} & \text{if } v \in S_i; \\ 0 & \text{o.w.} \end{cases}$$

Let H be an n by K matrix whose columns are the h_i, i.e. $H = [h_1, \cdots, h_K]$.

1. Show that $Ncut(S_1, \cdots, S_K) = Tr(H^T L H)$, where L is the Laplacian of G.
2. Show that $H^T D H = I_{K \times K}$
3. Show a spectral relaxation of minimum K-cut, generalizing Cheeger's rounding.