CS 598. Spectral Graph Theory
Problem Set 1

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Problem 1 *Courant-Fischer*

(5 pts.)

1. Prove the Eigenvalue Interlacing Theorem for decreasing order of eigenvalues (Lecture 3). Namely, show that if A is an n-by-n symmetric matrix and B a principal submatrix of A of dimension n-1 (that is, B is obtained by deleting the same row and column from A). Then $\alpha_1 \geq \beta_1 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1}$ Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of A and B respectively.

2. Show that if A is an n-by-n symmetric matrix and B be a principal submatrix of A of dimension n-k (that is, B is obtained by deleting the same set of k rows and columns from A). Then $\alpha_i \geq \beta_i \geq \alpha_{i+k}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of A and B respectively.

3. With the same assumptions as in item (2), show that if instead, we order the eigenvalues of A and B in increasing order, that is $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-k}$ then $\alpha_i \leq \beta_i \leq \alpha_{i+k}$.

Problem 2 *Independent Sets and Chromatic Number*

(5pts.)

Let G be a connected graph with average degree d_{av} and with Laplacian eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

1. Let S be an independent set in G and Let $d_{av}(S)$ be the average degree of a vertex in S. Show that
$$|S| \leq n\big(1 - \frac{d_{av}(S)}{\lambda_n}\big)$$

2. Let χ_G be the chromatic number of G. Show that
$$\chi_G \geq \frac{\lambda_n}{\lambda_n - d_{av}}$$

Problem 3 *Find the Spectrum*

(10pts.)

Let G be the following graph on $n = 2k + s$ vertices. Assume the vertices are numbered $\{1, \cdots, 2k, \cdots, 2k + s = n\}$. G has edges between all vertices $\{1, \cdots, k\}$ (a.k.a. there is a clique between the first k vertices), edges between all vertices $\{k + 1, \cdots, 2k\}$ (a.k.a. there is a clique between the second set of k vertices) and edges between every vertex in $\{2k + 1, \cdots, n = 2k + s\}$ and every other vertex. Moreover, assume there is a self-loop on every vertex. Let A be the adjacency matrix of G. Answer (with proof) the following questions:

1. What is the rank of matrix A?

2. Is A positive semidefinite?

3. Express A in the form $A = U^T \Sigma V$, where U and V are $n \times n$ orthonormal matrices and Σ is an $n \times n$ diagonal matrix with non-negative entries in the diagonal. The diagonal entries σ_i of Σ are also called singular values of A and this decomposition is called singular-value decomposition (SVD).

4. Find the eigenvalues and eigenvectors of A as a function of k, s. (If you can't find the eigenvalues exactly, then give upper-bounds and lowerbounds for each one of them).

Problem 4 *Upper bounds on Eigenvalues and Graphic Inequalitites*

(5 pts.)

1. For d a positive integer and $n = 2^d$, let G_n be the graph with vertex set $\{0, 1\}^d$ in which every pair of vertices that differ in at most two coordinates are joined by an edge. Prove upper and lower bounds on $\lambda_2(G_n)$. Make them as close to each other as possible.

2. For the complete binary tree T_n, prove that $\lambda_2(T_n) \geq 1/cn$ for some absolute constant c.

3. Let $w_1, \cdots, w_{n-1} > 0$ and let P be a weighted path with Laplacian $L_P = \sum_{i=1}^{n-1} w_i L_{(i,i+1)}$. Let v be any vector such that

   $$v(1) < v(2) < \cdots < v(n)$$

   Prove that

   $$\lambda_2(P) \geq min_{i:v_i \neq 0} \frac{(L_P v)_i}{v_i}$$

   This shows that a test vector can actually be used to prove a lower bound on λ_2!!