$x \in \mathbb{R} \implies x^2 \geq 0$

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But how does this help the cause of NP-Hard questions?

It so happens that a large (most?) number of NP-Hard questions can be formulated as a polynomial optimization questions!

For example, asking if the MaxCut of a graph is upper bounded by a constant c is equivalent to asking the following question,

For $f(x) = \sum_{(ij)\in E} \frac{(x_i - x_j)^2}{4}$, is $c - f(x) \geq 0$ over $x \in \{1, -1\}^{|V|}$?

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Definitions

- If one can write a $f \geq 0$ function over reals as $f = \sum_i^k g_i^2$ for polynomials g_i over reals such that $deg(g_i) \leq \frac{d}{2}$ then one is said to have found a *degree-d sum of squares certificate* for the non-negativity of f.

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- The "SOS degree" of a positive semidefinite function is the minimum d for which one can find a degree d SOS certification for it.

Hope and despair

Two famous theorems which justify the search for such certificates are,

What one can find...

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Unique Games Conjecture

In light of this, we can go back to the original MaxCut example to say that, what the "Unique Games Conjecture" implies is that there exists a graph G such that $\forall \epsilon > 0$, $MaxCut(G) - (0.878 + \epsilon)f(x)$ is a non-neative function with a SOS degree $\geq n^{\Omega(1)}$ (and hence can't be certified in polynomial time)

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The original formulation of the "Small Set Expansion Hypothesis" (SSEH) states that, (analogously to the Unique Games Conjecture),

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Famously in 2010, Raghavendra, Steurer and Tulsiani have shown a polynomial reduction from SSEH to UGC (proving the converse would be a breakthrough!) So we now focus on SOS methods to certify small set expansion properties!

Bounded hypercontractivity $\Longrightarrow$ expansion guarantees

We would like to call a $d-$regular graph $G = (V, E)$ a "small-set expander" if for any subset $S \subseteq V$ such that $\mu(S) \leq \delta$ it would imply that $\phi(S) \geq (1 - \delta)d$.

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For every even $q > 2$ a graph G is a small-set expander if and only if for every vector w in the large adjacency eigenvalue spaces of G it holds that $\mathbb{E}_i w_i^q \leq O((\mathbb{E}_i w_i^2)^{\frac{q}{2}})$

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$\phi(S) = \frac{nE(S,\bar{S})}{dmin\{|S|,|\bar{S}|\}} = \frac{\langle x,(L=I-\frac{A}{d})x\rangle_{l_2}}{\|x\|_{l_2}^2}$ for $|S| \leq \frac{n}{2}$.

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For any eigenvalue $\lambda \in (0,1)$ of $L = I - \frac{A}{d}$, let W be the $span\{v_i | Lv_i = (\leq \lambda)v_i\}$. If every $w \in W$ satisfies $\mathbb{E}_i w_i^4 \leq C(\mathbb{E}_i w_i^2)^2)$ then for all $S \subseteq V$ s.t $\mu(S) \leq \delta$ we have $\phi(S) \geq \lambda(1 - \sqrt{C\delta})$

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The above implies that if w is the characteristic vector of the set S then $\mu \geq \frac{1}{C}$. So for a "small set" is one with $\delta < \frac{1}{C}$ and hence it has expansion at least $\lambda - o(1)$

Introduction to expectation norms

We shall often be using the "expectation norm" or the "$L_p - norm$ of a random variable" X which is to be defined as, $\|X\|_p = (\mathbb{E}[|X|^p])^{1/p}$ for $p \geq 1$. This comes with its associated "expectation inner-product" defined as $| < f, g > | = \mathbb{E}[fg]$.

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1. For $1 < p, q < \infty$ satisfying the constraint $1/p + 1/q = 1$ and for X, Y random variables such that $\mathbb{E}[|X|^p], \mathbb{E}[|X|^q] < \infty$ we have,

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2. $\|X\|_q = max_{\|Y\|_{q/(q-1)} \leq 1} |\langle X, Y \rangle|$

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So for $\mu(S) = \delta \leq \frac{1}{2}$, over $x \in \{0,1\}^{|V|}$ we have,

$$\phi(S) = \frac{nE(S,\bar{S})}{dmin\{|S|,|\bar{S}|\}} = \frac{\langle x, (L = I - \frac{A}{d})x \rangle_{l_2}}{\|x\|_{l_2}^2}$$

$$\implies \phi(S) = \frac{\langle x, Lx \rangle}{\|x\|_2^2 = \delta}$$

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We decompose $x = x' + x''$ where $x' \in W$ and $x'' \in W^{\perp}$ and then we have the following inequality by using orthogonality and Holder's inequality (with $p = 4$ as defined in the previous slide),

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Further,

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But why is this expansion guarantee from the hypercontractivity assumption helpful? That is because in the important case of the Boolean hypercube this condition is easy to check!

For the Boolean hypercube $\{\pm\}^n$ one can write the eigenvectors as the functions, $\{\chi_S\}_{S\subseteq[n]}$. And these functions χ_S are defined on the vertices $x \in \{\pm 1\}^n$ as $\chi_S(x) = \prod_{i\in S} x_i$. And the eigenvalue of χ_S is $|S|/n$.

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Hence for any λ the subspace spanned by the eigenvectors of eigenvalue at most λ are is the subspace of $\{f : \{\pm 1\}^n \to \mathbb{R}\}$ which are spanned by $\chi_S(x) = \prod_{i \in S} x_i$ such that $deg(\chi_S) = |S| \leq \lambda n$.

This motivates the definition of the "k-junta" polynomials.

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$$f = \sum_{\alpha \subseteq [n]} \hat{f}_\alpha \chi_\alpha \to f' = \sum_{|\alpha| \leq d} \hat{f}_\alpha \chi_\alpha$$

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A "$n-$variate Fourier polynomial" with degree at most d is a function $f : \{\pm\}^n \to \mathbb{R}$ of the form, $f = \sum_{\alpha \subseteq [n], |\alpha| \leq d} \hat{f}_\alpha \chi_\alpha$

Theorem

Over the space of $n-$variate Fourier polynomials f with degree at most d, $\mathbb{E}[f^4] \leq 9^d(\mathbb{E}[f^2])^2$

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Theorem

If f and g are $n-$variate Fourier polynomials with degrees atmost d and e then it holds that, $\mathbb{E}[f^2 g^2] \leq 9^{\frac{d+e}{2}}(\mathbb{E}[f^2])(\mathbb{E}[g^2])$

Proof

Case 1

If one of the polynomials is a constant i.e d or $e = 0$ then it trivially follows from the implied independence that $\mathbb{E}[f^2 g^2] = \mathbb{E}(f^2)\mathbb{E}(g^2)$

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So rearranging the RHS (using $\mathbb{E}[x_n^{odd}] = 0$) we have,

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3. For an automorphism A of a vector space V we define its $p \to q$, "hypercontractive norm" as $\|A\|_{p \to q} = max_{v \in V} \frac{\|Av\|_q}{\|v\|_p}$. Hence we have effectively shown that there is an efficient SOS certificate for the hypercontractive norm bound, $\|\mathcal{P}\|_{p \to q} \leq 9^d$

Be wise. Generalize!

We can similarly define the projector $\mathcal{P}_{\geq \lambda}(G)$ into the subspace of a $d-$regular graph where adjacency eigenvectors are all atleast λ. Then using very similar techniques as above we can show that,

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For all $\epsilon, \delta \, 0, \|\mathcal{P}_{\geq \lambda}(G)\|_{2 \rightarrow q} \leq \frac{\epsilon}{\delta^{\frac{(q-2)}{2q}}}$ implies $\phi(|S| \leq \delta) \geq 1 - \lambda - \epsilon^2$

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Expansion implies norm bound!

There are constants $c_1, c_2 > 0$ such that for all $\delta > 0$,
$\Phi(|S| \leq \delta) > 1 - c_1(\frac{\lambda^2}{2^{c_2}})^q$ implies $\|\mathcal{P}\|_{2\rightarrow q} \leq \frac{2}{\sqrt{\delta}}$

But this is a much longer proof to present right now..