Resistance in networks and motivation from Physics.
- Ohm, Kirchoff and Laplacian.
- Effective Resistance.
- Linear Equations in Laplacians.
Resistor Networks

- One of our main motivations for studying the Laplacian matrix is its role in the analysis of physical systems.
- Given graph, we treat each edge as resistor.
- If graph unweighted, resistance is 1.
- Generally, resistance is $\frac{1}{w(e)}$.
- The smaller the weight, the larger resistance (if no edge, the infinite resistance).
Resistor Networks

- **Ohm’s Law**: Potential drop across a resistor is equal to the current flowing over the resistor times the resistance
  \[ V = IR \]

- For graph, define for each edge \((a,b)\) the current flowing from \(a\) to \(b\) as
  \[ i(a, b) = -i(b, a) \]

- If \(v \in R^n\) is the vector of potentials, then
  \[ i(a, b) = w(a, b) \cdot (v(a) - v(b)) \] (flowing from high to low potential)
Resistor Networks

• We want to write the equation in matrix form.
• Treat \( i \) as a vector with entries for every edge \((a,b)\) where \(a<b\).
• Recall the edge-vertex adjacency matrix

\[
U(((a, b), c)) = \begin{cases} 
1 & \text{if } a = c \\
-1 & \text{if } b = c \\
0 & \text{o.w}
\end{cases}
\]

• If \( W \) is the matrix with weights of edges in diagonal, then \( i = WUUv \).
Resistor Networks

- **Kirchoff’s Law**: Resistor networks cannot hold current. All flow entering from vertex $a$ from edges in the graph must exit $a$ to an external source.

- Let $i_{ext} \in \mathbb{R}^n$ denote the external currents, where $i_{ext}(a)$ is the amount of current entering the graph through node $a$.

- $i_{ext}(a) = \sum_{b:\{(a,b)\in E\}} i(a, b)$
Resistor Networks

- In matrix form
  \[ i_{ext} = U^T i = U^T WU v = L v \]
- External nodes when \( i_{ext} \neq 0 \), internal nodes when its zero.
- For internal nodes and unweighted graph, this equation implies:

\[
0 = L(a, \cdot)v = \\
\sum_{(a,b) \in E}(v(a) - v(b)) = d(a)v(a) - \sum v(b)
\]

- So voltages are weighted average of neighbors.
Resistor Networks

- We would like to apply the equation in reverse, and find the potentials from the currents.

\[ i_{ext} = L \nu = L^{-1} i_{ext} \Rightarrow \nu \]

- Laplacian has no inverse!!
- Define Pseudoinverse, since we are only interested in currents that sum to zero.
Resistor Networks

- **Definition** (Pseudoinverse).
  \[ LL^+ = \Pi \]
  Where \( \Pi \) is symmetric projection onto span of \( L \).

- **Claim.** \( L^+ = \sum_{i>1} \frac{1}{\lambda_i} \nu_i \nu_i^T \)

- In general, for every rational function of the laplacian...
The Path Graph

- Example: the path with unit flow.
Effective Resistance

- Effective resistance between vertices a and b is the resistance between a and b given by the whole network, if we treat it as a resistor.
- Recall serial and parallel composition.

\[ i(a, b) = \frac{v(a) - v(b)}{r_{a,b}} \]

Define \( R_{eff}(a, b) \) to be the potential difference between a and b if we send one unit of current through a and remove it from b.
Effective Resistance

- Define
  \[ i_{\text{ext}}(c) = \begin{cases} 
  1 & \text{if } c = a \\
  -1 & \text{if } c = b \\
  0 & \text{o.w} 
  \end{cases} \]

- This corresponds to a flow of 1 from a to b.
- Solve for voltage as before: \( L\nu = i_{\text{ext}} \Rightarrow \nu = L^+ i_{\text{ext}} \Rightarrow \nu(a) - \nu(b) = i_{\text{ext}}^T \nu = i_{\text{ext}}^T L^+ i_{\text{ext}}, \) the effective resistance.
- We can shift by multiple of all-1’s vector.
Effective Resistance

- **Claim.** $R_{\text{eff}}(a, b) = i^T_{\text{ext}} L^+ i_{\text{ext}} = v^T L v$

- **Claim.** If $a, b$ the external nodes and every other node $c$ is internal, then $v(a) \geq v(c) \geq v(b)$.

- Using this claim, show triangle inequalities. Effective Resistance is a distance.
Fixing Potentials

- We can see effective resistance by fixing potentials.
- Solve linear equations in Laplacians to find the potentials.
- Rest of class: properties of submatrices of Laplacian (see board).
Positive Inverse

• We will show the following claims:

• Claim. Let $A$ be a symmetric matrix with evals $-1 < \mu_1 \leq \cdots \leq \mu_n < 1$. Then

\[(I - A)^{-1} = \sum_j A^j\]
We will show the following claims:

**Claim.** Let $L = D - A$ the Laplacian of a connected graph. Let $X$ be a diagonal, non-negative, non-zero matrix. Then
\[
D - A + X \text{ is positive definite}
\]
\[
D + A + X \text{ is positive definite}
\]
\[
(L + X)_{ij}^{-1} > 0 \text{ for all } i,j.
\]