CS 598: Spectral Graph Theory. Lecture 12

A construction of Expanders

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Today

- Deterministic construction of expanders
- Start from small graph, what to do next?
- Squaring the graph
- Replacement product
- A construction of $1/10$-expanders.
Operations on a Graph

- We saw last time that we can start from a small good expander (can find one easily), perform some operation and get a bigger expander. Few ways to do it:
  - Lifts (in two weeks)
  - Squaring
  - Replacement Products
  - Combinations of the above
**Attempt 1: Squaring a Graph**

- We can improve the expansion of a graph by squaring it.
- So we could start from large non-expanding graph and get expander perhaps? 😊

**Definition:** We define the square of a graph $G^2$ to be a the graph with adjacency matrix $A_{G^2} = A_G^2 - dI_n$

- $G^2$ has an edge between $(u,v)$ if $(u,v)$ are connected with path of length two. Remove self-loops.
- If graph has no 4-cycles, then $A_G^2(u,v) = 1$ for such two nodes.
Theorem:
Let $G$ be a $d$-regular graph with Laplacian eigenvalues $\lambda_1, \ldots, \lambda_n$. Then, $G^2$ is a $d(d-1)$ regular graph with Laplacian eigenvalues $2d\lambda_i - \lambda_i^2$. In particular, the largest eigenvalue is at most $d^2$.

- If $G$ is $\epsilon$-expander, then ratio of eigenvalues to degree is $(1 \pm \epsilon)$.
- Squaring the graph, we get ratio of eigenvalues to degree $(1 \pm \epsilon^2)$, so it is an $\epsilon^2$-expander.
Squaring a Weak Expander

- By previous theorem we can see that if a graph is a weak expander in the sense $\lambda_2 = \delta d$, where $\delta \ll 1$, then the expansion almost doubles by squaring.

- The degree also squares –not so useful 😞
Attempt 2: Line Graph

- Line graph $H$ of $G$ has a vertex per edge of $G$, where two are connected if they share an endpoint in $G$.
- $G$ d-regular has $nd/2$ edges, so $H$ has that many vertices.
- $H$ has degree $2(d-1)$
- If we consider one vertex $u$ of $G$, then all the $d$ edges $(u,v)$ will be connected in $H$: They form a d-clique!
- Every $u$ belongs to two d-cliques.
Spectrum of Line Graph

- **Theorem.**

Let $G$ be a $d$-regular graph with $n$ vertices and $H$ be the line graph. Then the spectrum of the Laplacian of $H$ is the same as the spectrum of the Laplacian of $G$, except that it has \( \frac{nd}{2} - n \) extra eigenvalues equal to $2d$. 
Attempt 2: Line Graph

- Line graph is bigger than G, and degree is only a factor of two bigger, so we are in better shape, but still need to keep degree down 😞

- Measure the quality of expansion with spectral ratio: \( r(G) = \min \left( \frac{\lambda_2}{d}, \frac{2d - \lambda_n}{d} \right) \)
  - If G is a-expander then \( r(G) > 1 - a \).
  - The closer to 1 (bigger) \( r(G) \) is, the better the expansion.
Spectrum of Line Graph

- **Theorem.**

Let $G$ be a $d$-regular graph for $d > 5$ and let $H$ be its line graph. Then

$$r(H) = \frac{\lambda_2(G)}{2(d - 1)} \geq \frac{r(G)}{2}$$
Plan of Attack

- We see that line graph has half the spectral radius of $G$. But also has more vertices.
- Plan is to build an infinite family of $d$-regular graphs with spectral ratio bounded below by some absolute constant $b$:
  - Begin with small expander.
  - Take the line graph (Increase size)
  - Replace the cliques with expanders (decrease degree).
  - Square (improve expansion).
  - Repeat.
Approximation of Line Graph

- In order to increase degree, we approximate line graph.
- We know how to approximate cliques with expanders.
- Let’s replace every d-clique in the line graph of G with an expander Z of d nodes and degree z call this new graph $G_{OLZ}$.
Approximation of Line Graph

• **Theorem.** Let $G$ be a $d$-regular graph. Let $H$ be the line graph of $G$ and let $Z$ be a $z$-regular $\epsilon$ — expander. Then

$$ (1 - \epsilon) \frac{z}{d} H \preceq Go_L Z \preceq (1 + \epsilon) \frac{z}{d} H $$
Approximation of Line Graph

- Corrolary.

\[ r(G_{o_L}Z) \geq \frac{(1-\epsilon)}{2} r(G) \]

- Now can we put all of those things together to get d-regular expanders?
Putting it Together

- Start from $G_0$ a $d$-regular graph, with spectral ratio $\beta = \frac{1}{5}$ (say). Also take $Z$ to be a $z$-regular $\epsilon$-expander on $d$ nodes where $d = \left(2z(2z - 1)\right)^2 - 2z(2z - 1)$. Take $\epsilon = \frac{1}{6}$ (say).
- Goal is to create arbitrarily large $d$-regular graphs of same spectral ratio (or better).
Putting it Together

- Construct $G_{o_L}Z$. Degree is 2z, spectral radius about $\beta/2$. Need to increase spectral radius (need it to be bigger than $\beta$ for every size graph), so square twice.
- Let $G_1 = ((G_{o_L}Z)^2)^2$
- Need Z of degree z and number of nodes $d = (2z(2z - 1))^2 - 2z(2z - 1)$ as taken.
- Repeat, and set $G_i = ((G_{i-1}o_LZ)^2)^2$