Extremal Eigenvalues and Eigenvectors of the Laplacian and the Adjacency Matrix.
Today

- Bounding eigenvalues
- Courant-Fischer and Rayleigh quotients
- Applications of Courant-Fischer
- Adjacency matrix vs. Laplacian
- Chromatic number
- Perron-Frobenius
Start Bounding Laplacian Eigenvalues
Sum of Eigenvalues, Extremal Eigenvalues

- \( \sum_i \lambda_i = \sum_i d_i \leq d_{max} n \) where \( d_i \) is the degree of vertex \( i \).
  
  Proof: take the trace of \( L \)

- \( \lambda_2 \leq \frac{\sum_i d_i}{n-1} \) and \( \lambda_n \geq \frac{\sum_i d_i}{n-1} \)
  
  Proof: previous inequality + \( \lambda_1 = 0 \).
For any nxn symmetric matrix $A$ with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \ldots, v_n$, denote $S_k$ the span of $v_1, v_2, \ldots, v_k$ and $S_k^\perp$ the orthogonal complement, then

$$\alpha_k = \max_{x \in S_{k-1}^\perp, x \neq 0} \frac{x^T Ax}{x^T x} \quad \alpha_1 = \max_{x \neq 0} \frac{x^T Ax}{x^T x}$$

Proof: see blackboard
Courant-Fischer

- Courant-Fischer Min Max Formula: For any nxn symmetric matrix $A$ with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order),

\[
\alpha_k = \max_{S \subseteq \mathbb{R}^n, \dim(S) = k} \min_{x \in S} \frac{x^T Ax}{x^T x}
\]

\[
\alpha_k = \min_{S \subseteq \mathbb{R}^n, \dim(S) = n-k+1} \max_{x \in S} \frac{x^T Ax}{x^T x}
\]

Proof: see blackboard
Courant-Fischer for Laplacian

- Courant-Fischer Min Max Formula for increasing eigenvalue order (e.g. Laplacians): For any nxn symmetric matrix L, with eigenvalues in increasing order

\[ \lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T L x}{x^T x} \]

\[ \lambda_k = \max_{S \text{ of dim } n-k-1} \min_{x \in S} \frac{x^T L x}{x^T x} \]

- Definition (Rayleigh Quotient): The ratio \( \frac{x^T L x}{x^T x} \) is called the Rayleigh Quotient of x with respect to L.

- We will use it to bound eigenvalues of Laplacians of certain graphs.
Courant-Fischer for Laplacian

- Applying Courant-Fischer for the Laplacian we get:

\[ \lambda_1 = 0, \nu_1 = 1 \]

\[ \lambda_2 = \min_{x \perp 1, x \neq 0} \frac{x^T L x}{x^T x} = \min_{x \perp 1, x \neq 0} \frac{\sum (x_i - x_j)^2}{\sum x_i^2} \]

\[ \lambda_{\text{max}} = \max_{x \neq 0} \frac{x^T L x}{x^T x} = \max_{x \neq 0} \frac{\sum (x_i - x_j)^2}{\sum x_i^2} \]

- Useful for getting bounds, if calculating spectra is cumbersome.
- To get upper bound on \( \lambda_2 \), just need to produce vector with small Rayleigh Quotient.
- Similarly, to get lower bound on \( \lambda_{\text{max}} \), just need to produce vector with large Rayleigh Quotient.
**Example 1**

- *Lemma 1*: Let $G=(V,E)$ be a graph with some vertex $w$ having degree $d$. Then
  \[ \lambda_{\text{max}} \geq d \]

- *Lemma 2*: We can also improve on that. Under same assumptions, we can show:
  \[ \lambda_{\text{max}} \geq d + 1 \]

Proof: see blackboard
Example 1

- **Lemma 1:** Let $G=(V,E)$ be a graph with some vertex $w$ having degree $d$. Then

$$\lambda_{\text{max}} \geq d$$

- **Lemma 2:** We can also improve on that. Under same assumptions, we can show:

$$\lambda_{\text{max}} \geq d + 1$$

Lemma 2 is tight, take star graph (ex)
Example 2

- The Path graph $P_n$ on $n$ vertices has

$$\lambda_2 \leq \frac{12}{n^2}$$

- Already knew that eigenvalues are

$$2 - 2 \cos \left(\frac{\pi k}{n}\right) \approx 2 \left(1 - 1 + \frac{\pi^2}{2n^2}\right) \approx \frac{\pi^2}{n^2},$$

but this is easier and more general.

Proof: see blackboard
Example 3

- The complete binary tree \( B_n \) on \( n = 2^d - 1 \) vertices has

\[ \lambda_2 \leq \frac{2}{n - 1} \]

\( B_n \) is the graph with edges of the form \( (u, 2u) \) and \( (u, 2u+1) \) for \( u < n/2 \).

Proof: See blackboard
Example 3
• Lower bounds are harder, we will see some in two lectures (different technique)
Adjacency Matrix vs. Laplacian
Adjacency Matrix Refresher

G = \{V,E\}

- Unweighted graphs for simplicity

A has n eigenvalues (counting multiplicities)
\{a_1 \geq a_2 \geq ... \geq a_n\}

- Adjacency matrix as operator:

\[(A_G u)(i) = \sum_{j:(i,j) \in E} v(j)\]
Adjacency Matrix vs. Laplacian for d-regular graphs

- G is d-regular if every vertex has degree d. In this case: $L_G = D_G - A_G = dI - A_G$
- Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $L$ and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ be the eigenvalues of $A$.
- We have $\alpha_i = d - \lambda_i$ and the corresponding eigenvectors are the same.
Bounds on the Eigenvalues of Adjacency Matrix

- $\alpha_1 \leq d_{\text{max}}$

Proof: See blackboard.

- Adjacency matrix as operator:
  $$(A_G \mathbf{u})(i) = \sum_{j: (i,j) \in E} \mathbf{v}(j)$$
Bounds on the Eigenvalues of Adjacency Matrix

- \( \alpha_1 \leq d_{\text{max}} \) with equality iff graph is \( d_{\text{max}} \)–regular. In this case, the first eigenvector is the all-one’s vector. (exercise)
Will see next how to apply Courant-Fischer for the adjacency matrix to get another bound on the first eigenvalue as well as a relation to graph coloring.
Bounding Adjacency Matrix

Eigenvalues

 Lemma 1: $\alpha_1$ is at least the average degree of the vertices in $G$

Proof: see blackboard

$$\alpha_1 = \max_{x \neq 0} \frac{x^T Ax}{x^T x}$$
Lemma 1: $\alpha_1$ is at least the average degree of the vertices in $G$.

While we may think of $\alpha_1$ as being related to the average degree, it behaves differently. If we remove the vertex of smallest degree in a graph, the average degree can increase. However, $\alpha_1$ only decreases when we remove a vertex.

Lemma 2: Let $A$ be a symmetric matrix, let $B$ be the matrix obtained by removing the last row and column from $A$ and let $b_1$ be the largest eigenvalue of $B$. Then $\alpha_1 \geq b_1$.

Proof: see blackboard
The chromatic number of a graph $G$, denoted $\chi(G)$, is the least $k$ for which $G$ has a $k$-coloring.

**Theorem (Wilf):** $\chi(G) \leq \lceil \alpha_1 \rceil + 1$

Proof: see blackboard
Chromatic Number

- The chromatic number of a graph $G$, denoted $\chi(G)$, is the least $k$ for which $G$ has a $k$-coloring.

- **Theorem (Wilf):** $\chi(G) \leq \lceil \alpha_1 \rceil + 1$

- Improvement over classical bound $\chi(G) \leq d_{\text{max}} + 1$, as there are graphs (e.g. path graph) where $\alpha_1$ is much less than $d_{\text{max}}$
We saw what happens for regular graphs. What is $G$ is not regular? We know that $\alpha_1 < d_{\text{max}}$ but what about $v_1$?

**Perron-Frobenius Theorem (for graphs):** Let $G=(V,E,w)$ be a connected graph, $A$ its adjacency matrix and $\mu_1 \geq \cdots \geq \mu_n$ its evales. Then:

1. $\mu_1 \geq -\mu_n$
2. $\mu_1 > \mu_2$
3. $\mu_1$ has a strictly positive eigenvector
Bipartite?

• Another graph Property and eigenvalues: Bipartiteness

• **Theorem**: If $G$ is a connected graph then $\mu_1 = -\mu_n$ iff $G$ is bipartite.
The most negative eigenvalue of the adjacency matrix (and the largest eigenvalue of the Laplacian) corresponds to the highest frequency vibration in a graph. Its eigenvector tries to assign as different as possible values to neighbors. Corresponds to coloring.
Theorem (Hoffman). Let $S$ be an independent set in $G$, and let $d_{av}(S)$ be the average degree of a vertex in $S$. Then

$$|S| \leq n \left(1 - \frac{d_{av}(S)}{\lambda_n}\right)$$

It follows that $\chi_G \geq \frac{\lambda_n}{\lambda_n - d_{av}}$ (exercise)
Laplacian: The Perron-Frobenius Theorem

- Theory can also be applied to Laplacians and any matrix with non-positive off-diagonal entries. It involves the eigenvector with smallest eigenvalue.

**Perron-Frobenius for Laplacians:** Let M be a matrix with non-positive off-diagonal entries s.t. the graph of the no-zero off-diagonal entries is connected. Then the smallest eigenvalue has multiplicity 1 and the corresponding eigenvector is strictly positive.

Next time we will see how to apply Peron Frobenius to show Fiedler’s nodal domain theorem.