Problem 1  Courant-Fischer

(5 pts.)

1. Prove the Eigenvalue Interlacing Theorem for decreasing order of eigenvalues (Lecture 3). Namely, show that if $A$ is an $n$-by-$n$ symmetric matrix and $B$ a principal submatrix of $A$ of dimension $n-1$ (that is, $B$ is obtained by deleting the same row and column from $A$). Then $\alpha_1 \geq \beta_1 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1}$ Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of $A$ and $B$ respectively.

2. Show that if $A$ is an $n$-by-$n$ symmetric matrix and $B$ be a principal submatrix of $A$ of dimension $n-k$ (that is, $B$ is obtained by deleting the same set of $k$ rows and columns from $A$). Then $\alpha_i \geq \beta_i \geq \alpha_i + k$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of $A$ and $B$ respectively.

3. With the same assumptions as in item (2), show that if instead, we order the eigenvalues of $A$ and $B$ in increasing order, that is $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-k}$ then $\alpha_i \leq \beta_i \leq \alpha_i + k$.

Problem 2  Independent Sets and Chromatic Number

(5pts.)

Let $G$ be a connected graph with average degree $d_{av}$ and with Laplacian eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

1. Let $S$ be an independent set in $G$ and Let $d_{av}(S)$ be the average degree of a vertex in $S$. Show that

$$|S| \leq n \left(1 - \frac{d_{av}(S)}{\lambda_n}\right)$$

2. Let $\chi_G$ be the chromatic number of $G$. Show that

$$\chi_G \geq \frac{\lambda_n}{\lambda_n - d_{av}}$$
Problem 3  Find the Spectrum

(10pts.)

Let $G$ be the following graph on $n = 2k + s$ vertices. Assume the vertices are numbered $\{1, \cdots, 2k, \cdots, 2k + s = n\}$. $G$ has edges between all vertices $\{1, \cdots, k\}$ (a.k.a. there is a clique between the first $k$ vertices), edges between all vertices $\{k + 1, \cdots, 2k\}$ (a.k.a. there is a clique between the second set of $k$ vertices) and edges between every vertex in $\{2k + 1, \cdots, n = 2k + s\}$ and every other vertex. Moreover, assume there is a self-loop on every vertex. Let $A$ be the adjacency matrix of $G$. Answer (with proof) the following questions:

1. What is the rank of matrix $A$?
2. Is $A$ positive semidefinite?
3. Express $A$ in the form $A = U^T \Sigma V$, where $U$ and $V$ are $n \times n$ orthonormal matrices and $\Sigma$ is an $n \times n$ diagonal matrix with non-negative entries in the diagonal. The diagonal entries $\sigma_i$ of $\Sigma$ are also called singular values of $A$ and this decomposition is called singular-value decomposition (SVD).
4. Find the eigenvalues and eigenvectors of $A$ as a function of $k, s$. (If you can’t find the eigenvalues exactly, then give upper-bounds and lowerbounds for each one of them).

Problem 4  Upper bounds on Eigenvalues and Graphic Inequalitities

(5 pts.)

1. For $d$ a positive integer and $n = 2^d$, let $G_n$ be the graph with vertex set $\{0, 1\}^d$ in which every pair of vertices that differ in at most two coordinates are joined by an edge. Prove upper and lower bounds on $\lambda_2(G_n)$. Make them as close to each other as possible.

2. For the complete binary tree $T_n$, prove that $\lambda_2(T_n) \geq 1/cn$ for some absolute constant $c$.

3. Let $w_1, \cdots, w_{n-1} > 0$ and let $P$ be a weighted path with Laplacian $L_P = \sum_{i=1}^{n-1} w_i L_{(i,i+1)}$. Let $v$ be any vector such that $v(1) < v(2) < \cdots < v(n)$

Prove that

$$\lambda_2(P) \geq \min_{i \neq 0} \left( \frac{(L_P v)_i}{v_i} \right)$$

This shows that a test vector can actually be used to prove a lower bound on $\lambda_2$!!