CS 598RM: Algorithmic Game Theory, Spring 2017
HW 4 (extra homework). Due on Monday, May 1st at 11:59pm CST.

Instructions:

1. We will grade this assignment out of a total of 30 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whatever is suitable. See instructions on the webpage on how to submit.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

This is on market equilibrium. Recall that in a Fisher market, there is a set B of buyers, and a set G of goods. Let $m \stackrel{\text{def}}{=} |B|$ and $n \stackrel{\text{def}}{=} |G|$. In a general Fisher market, preferences of buyer $i \in B$ are represented by a non-decreasing, non-negative, concave utility function $U_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$, and she has money endowment e_i to buy what she likes. Available quantity of good j is q_j. Let us also assume that function U_i is *non-satiated*, that is given utility of buyer i at any bundle there is another bundle where she gets strictly more utility. Formally, for each $i \in B$, given $\boldsymbol{x} \in \mathbb{R}^n_+$, $\exists \boldsymbol{y} \in \mathbb{R}^n_+$ such that $U_i(\boldsymbol{y}) > U_i(\boldsymbol{x})$.

Prices $\boldsymbol{p}^* = (p^*_1, \ldots, p^*_n) \geq 0$ and allocation $\boldsymbol{x}^* = (\boldsymbol{x}^*_1, \ldots, \boldsymbol{x}^*_m) \geq 0$ where x^*_{ij} is the amount of good j allocated to buyer i, are said to be at equilibrium if:

(1) $\boldsymbol{x}^*_i$ is an optimal bundle of buyer i at prices $\boldsymbol{p}^*$, *i.e.*, $\boldsymbol{x}^*_i \in \arg\max_{\boldsymbol{x}_i \geq 0;\ \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq e_i} U_i(\boldsymbol{x}_i)$

(2) Each buyer spends all its money, *i.e.*, $\boldsymbol{x}^*_i \cdot \boldsymbol{p}^* = e_i, \ \forall i \in B$,

(3) Demand does not exceed supply for any good, and for goods with positive price demand equals supply, *i.e.*, $\forall j \in G, \ \sum_{i \in B} x^*_{ij} \leq q_j$, and if $p_j > 0$ then $\sum_{i \in B} x^*_{ij} = q_j$.

1. (2 points) Show that it is without loss of generality to assume that $q_j = 1,\ \forall j \in G$. That is given a market with arbitrary q_js reduce it to another market with unit quantities for each good whose equilibria can be mapped back to the equilibria of the original market.

2. (13 points) Consider Leontief utility function (discussed briefly in the second lecture on markets), where goods are complements like bike and helmet. Formally, utility function of buyer $i \in B$ is $U_i(\boldsymbol{x}_i) = \min_{j \in G} \frac{x_{ij}}{U_{ij}}$, where $U_{ij} \geq 0,\ \forall j$.

   Show that prices and allocation $(\boldsymbol{p}^*, \boldsymbol{x}^*)$ gives an equilibrium if and only if in addition to (2) and (3) above, it satisfies the following: $\forall i \in B,\ x^*_{ij} \geq \beta^*_i U_{ij}$ and if $p^*_j > 0$ then $x^*_{ij} = \beta^*_i U_{ij},\ \forall j \in G$, where $\beta^*_i = U_i(\boldsymbol{x}^*_i)$. The latter condition essentially captures (1).

   Using this show that following convex program and its dual variables give equilibrium utility and prices respectively.

$$\begin{array}{ll} \max: & \sum_{i \in B} e_i \log(\beta_i) \\ subjectto & \sum_{i \in B} U_{ij}\beta_i \leq 1, \quad \forall j \in G \\ & \beta_i \geq 0, \quad \forall i \in B \end{array}$$

3. (5 points) Construct a market with Leontief utilities, such that all U_{ij}s and e_is are rational numbers, while all equilibrium prices are irrational.

   (Hint: try and think of constructing two good three agent market.)

4. (10 points) Give a strongly polynomial time algorithm for the linear case (the one we discussed in class, where $U_i(\boldsymbol{x}_i) = \sum_{j \in G} U_{ij} x_{ij}$) under the assumption that all U_{ij}'s are 0/1 (Exercise 5.1 of the AGT book).

   An algorithm is strongly polynomial time if it's running time depends only on m and n, and not on (log of) e_i's and U_{ij}s.