CS 598RM: Algorithmic Game Theory, Spring 2017
HW 3 (due on Monday, April 17th at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 30 points.

2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.

3. Please type your solutions if possible in Latex or doc whatever is suitable. See instructions on the webpage on how to submit.

4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.

5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.

6. No late assignments will be accepted.

7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (5 points)

   - Prove that for every false bid $b_i \neq v_i$ by a bidder in a Vickrey auction, there exist bids b_{-i} by the other bidders such that i's payoff when bidding b_i is strictly less than when bidding v_i.

   - Consider a first price single item auction with set of bidders A. Let $n = |A|$, and private value v_i of each bidder i comes from a uniform distribution [0, 1]. Show that bid profile $(b_1, \ldots, b_n)$ such that $b_i = \frac{(n-1)}{n} v_i$, $\forall i \in A$ is a Nash equilibrium.
   Extra (No need to submit): What is the expected revenue of the auctioneer at this Nash equilibrium? How does it compare with the expected revenue of Vickrey Auction?

2. (15 points) Recall the sponsored search auction problem discussed in class: there are k slots, the jth slot has a known click-through rate (CTR) of α_j (non-increasing in j), and the payoff of bidder i in slot j is $\alpha_j(v_i - p_j)$, where v_i is the (private) value-per-click of the bidder and p_j is the price charged per-click in that slot. For historical reasons, modern search engines do not use the truthful auction discussed in class. Instead, they use auctions derived from the Generalized Second-Price (GSP) auction, defined as follows:

(1) Rank advertisers by bid; assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_n$.

(2) For $i = 1, 2, \ldots, k$, assign the ith bidder to the ith slot.

(3) For $i = 1, 2, \ldots, k$, charge the ith bidder a price of b_{i+1} per click. (if $n \leq k$, then the price charged to the bidder n is zero.)

(a) (2 points) Prove that for $k = 2$ and any sequence $\alpha_1 \geq \alpha_2 > 0$, there exist valuations for the bidders such that the GSP auction is not truthful. (This in fact holds for any $k \geq 2$).

(b) (3 points) Fix CTRs for slots and valuations-per-click for bidders. We can assume that $k = n$ by adding dummy slots with zero CTR (if $k < n$) or dummy bidders with zero valuation (if $k > n$). A bid vector b is an equilibrium of GSP if no bidder can increase its payoff by changing its bid. Verify that this translates to the following conditions, assuming that $b_1 \geq b_2 \geq \cdots \geq b_n$: for every i and higher slot $j < i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j);$$

and for every lower slot $j > i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}).$$

(Derive these by adopting i's perspective and "targeting" the slot j.)

(c) (5 points) A bid vector **b** with $b_1 \geq \cdots \geq b_n$ is envy-free if for every bidder i and slot $j \neq i$,

$$\alpha_i(v_i - p_i) \geq \alpha_j(v_i - p_j);$$

where $p_j = b_{j+1}$

Verify that an envy-free bid vector is necessarily an equilibrium. (The terminology "envy-free" stems from the following interpretation: for the current price-per-click is p_j of slot j; then the above inequalities say: "each bidder i is as happy getting its current slot at its current price as it would be getting any other slot and that slot's current price".)

(Hint: you might want to first prove that if the bid vector is envy-free, then the bidders must be sorted in non-increasing order of valuations.)

(d) (5 points) Prove that, for every set of α_js and v_is, there is an equilibrium of the GSP auction for which the outcome (i.e., the assignment of bidders to slots) and the prices paid precisely match those of the truthful auction discussed in class. If you want, you can assume that the CTRs are strictly decreasing.

(Hint: Recall that using Myerson's lemma (see notes suggested for Lecture 18) we know a closed-form solution for the payments made by the truthful auction. What bids would yield these payments in a GSP auction? Part (d) might be useful for proving that they form an equilibrium.)

3. (5 points) This problem considers variations on the Bulow-Klemperer Theorem (Lecture 21). Consider selling $k \geq 1$ identical items (with at most one given to each bidder) to bidders with valuations drawn iid from F, where F is a regular distribution (i.e., the corresponding virtual welfare function ϕ is a monotonically increasing function). Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Myerson's optimal auction for F with n bidders.

   (Hint: Myerson's optimal auction will be Vickrey with reserve $\phi^{-1}(0)$, *i.e.,* discard bids below $\phi^{-1}(0)$, give the item to k highest bidders and charge them $\max\{(k+1)^{th}\text{highest bid}, \phi^{-1}(0)\}$)

4. (5 points) Consider a combinatorial auction in a which a bidder can submit multiple bids under different names, unbeknownst to the mechanism. The allocation and payment of a bidder is respectively the union of the allocations and sum of the payments, respectively assigned to all of its pseudonyms. Show that the following is possible: a bidder in a combinatorial auction can earn higher utility from the VCG mechanism by submitting multiple bids than by bidding truthfully.

   Can this ever happen in the Vickrey auction? Give a brief explanation.

   The remaining problems are for self study. Do *NOT* submit for grading.

- Recall the Generalized Second Price (GSP) keyword auction from Problem 2. For this problem we will think about bidders with known valuations and the pure-strategy Nash equilibria of the corresponding game. Note that in part (e) of Problem 2 you proved that in every such game (for any k, n, valuations, and click-through rates), there is a pure-strategy Nash equilibrium with the maximum-possible social-welfare. That is, the best Nash equilibrium captures the full social-welfare. This problem investigates additional Nash equilibria that do not share this property.

  (a) Show that even when number of slots $k = 1$ and number of bidders $n = 2$, the price of anarchy of the GSP game can be arbitrarily bad, where
  $$\text{PoA} = \frac{\text{OPT social-welfare}}{\text{Social-welfare at worst Nash equilibrium}}.$$
  [Hint: Think of the case when bidders may bid more than their value, and think of the Nash equilibrium condition].

  (b) Consider now a Nash equilibrium in which every bid b_i is at most the player's valuation v_i. Suppose that players i and j are assigned to slots with click-through rates α_h and α_l respectively, with $h < l$. Prove that
  $$\frac{\alpha_l}{\alpha_h} + \frac{v_i}{v_j} \geq 1$$

- Consider a combinatorial auction on m goods where you know a priori that every bidder is unit demand. This means that the valuation of a bidder i can be described by m private parameters (one per good) $v_{i1}, \ldots, v_{im}$, and its valuation for an arbitrary set S of goods is defined as $max_{j \in S} v_{ij}$. Prove that the VCG mechanism can be implemented in polynomial time for unit-demand bidders.

- Consider the reverse auction we briefly talked about in class: A denotes the set of bidders who are willing to sell the spectrum they hold. There is a set $F \subseteq 2^B$ of feasible sets that is upward closed (i.e., supersets of feasible sets are again feasible), i.e., if $S \in F$ then we can repack S in available range if spectrum of $A \setminus S$ is acquired.

    - Initialize $S = A$.
    - While there is a bidder $i \in S$ such that $S \setminus \{i\}$ is feasible:
                (*) Delete some such bidder i from S such that $S \setminus \{i\}$ is feasible.
    - Return S.

Suppose we implement the step (*) using a scoring rule, which assigns a number to each bidder i. At each iteration, the bidder with the largest score (whose deletion does not destroy feasibility of S) gets deleted. The score assigned to a bidder can depend on i, is bid, the bids of other bidders that have already been deleted, the feasible set F, and the history of what happened in previous iterations. (Note a score is not allowed to depend on the value of the bids of other bidders that have not yet been deleted.) Assume that the scoring rule is increasing – holding everything fixed except for b_i, is score is increasing in its bid b_i. Then, show that the allocation rule above is monotone: for every i and b_{-i}, if i wins with bid