CS 598RM: Algorithmic Game Theory, Spring 2017
HW 2 (due on Monday, March 13th at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 30 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whatever is suitable. See instructions on the webpage on how to submit.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (5 points)
   - Give example of a two-player game with pure NE, where best response dynamics can cycle.
   - Give example of a potential game where the strategy profile achieving minimum potential does not give the minimum cost NE.

     [Hint: Try cost-sharing game.]
2. (5 points) Show that in every network cost-sharing game, the PoA is at most k, where k is the number of players.
3. (5 points) Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Piguo bound $\alpha(\mathcal{C})$ is 4/3.
4. (10 points) Consider an atomic selfish routing game in which all players have the same source vertex and sink vertex (and each controls one unit of flow). Assume that edge cost functions are non-decreasing, but do not assume that they are affine. Prove that a pure-strategy Nash equilibrium can be computed in polynomial time. Be sure to discuss the issue of fractional

vs. integral flows, and explain how (or if) you use the hypothesis that edge cost functions are non-decreasing.

[Hint: Recall the Rosenthal's potential function. You can assume without proof that the minimum-cost flow can be solved in polynomial time. If you haven't seen the min-cost flow problem before, you can read about it in any book on "combinatorial optimization".]

5. (5 points) This problem develops some theory about potential games; we talked about these while discussing selfish routing. We consider an abstract finite game with n players with finite strategy sets $S_1, \ldots, S_n$. Each player has a payoff function π_i mapping outcomes (elements of $S_1 \times \cdots \times S_n$) to real numbers. Recall that a potential function for such a game is defined by the following property: for every outcome $\boldsymbol{s} \in S_1 \times \cdots \times S_n$, every player i, and every deviation $s'_i \in S_i$.

$$\pi_i(s'_i, \boldsymbol{s}_{-i}) - \pi_i(s_i, \boldsymbol{s}_{-i}) = \Phi(s'_i, \boldsymbol{s}_{-i}) - \Phi(s_i, \boldsymbol{s}_{-i}).$$

A team game is a game in which all players have the same payoff function: $\pi_1(\boldsymbol{s}) = \cdots = \pi_n(\boldsymbol{s})$ for every outcome $\boldsymbol{s}$. In a dummy game, the payoff of every player i is independent of its strategy: $\pi_i(s_i, \boldsymbol{s}_{-i}) = \pi_i(s'_i, \boldsymbol{s}_{-i})$ for every $\boldsymbol{s}_{-i}$ and every $s_i, s'_i \in S_i$.

Prove that a game with payoffs $\pi_1, \ldots, \pi_n$ is a potential game (i.e., admits a potential function) if and only if it is the sum of a team game $\pi_1^t, \ldots, \pi_n^t$ and a dummy game $\pi_1^d, \ldots, \pi_n^d$ (*i.e.,* $\pi_i(\boldsymbol{s}) = \pi_i^t(\boldsymbol{s}) + \pi_i^d(\boldsymbol{s})$ for every i and $\boldsymbol{s}$.)

The remaining problems are for self study. Do *NOT* submit for grading.

- Prove that if a game admits two potential functions Φ_1 and Φ_2, then Φ_1 and Φ_2 differ by a constant. That is, for some $c \in \mathbb{R}$, $\Phi_1(\boldsymbol{s}) - \Phi_2(\boldsymbol{s}) = c$ for every outcome $\boldsymbol{s}$ of the game.

  Thus, it is well defined to speak of "the potential function maximizer" of a potential game.

- Prove that a finite game admits a potential function if and only if for every two outcomes $\boldsymbol{s}^1$ and $\boldsymbol{s}^2$ that differ in two players' choices, say players i and j,

$$(\pi_i(s_i^2, \boldsymbol{s}_{-i}^1) - \pi_i(\boldsymbol{s}^1)) + (\pi_j(\boldsymbol{s}^2) - \pi_j(s_i^2, \boldsymbol{s}_{-i}^1)) = (\pi_j(s_j^2, \boldsymbol{s}_{-j}^1) - \pi_j(\boldsymbol{s}^1)) + (\pi_i(\boldsymbol{s}^2) - \pi_i(s_j^2, \boldsymbol{s}_{-j}^1))$$

- As we discussed in class, a congestion game is like an atomic selfish routing game except we drop the assumption that strategies represent paths in a network. That is, there is a ground set E (previously, the edges), and each $e \in E$ has a cost function c_e. Each player i has a strategy set S_i, and each strategy $s_i \in S_i$ is a subset of E (previously, a path). In an outcome $\boldsymbol{s} = (s_1, \ldots, s_n)$, if x_e players are using a strategy that contains e, then player i's cost is $\sum_{e \in s_i} c_e(x_e)$. We showed in class that every congestion game is a potential game.

Next we prove the converse. Two games G_1 and G_2 are isomorphic if: (i) they have the same number k of players; (ii) for each i, there is a bijection f_i from the strategies A_i of player i in G_1 to the strategies B_i of player i in G_2; and (iii) these bijections preserve payoffs, so that $\pi_i^1(s_1, \ldots, s_n) = \pi_i^2(f_1(s_1), \ldots f_n(s_n))$ for every player i and outcome $(s_1, \ldots, s_n)$ of G_1. (Here π^1 and π^2 are the payoff functions of G_1 and G_2 respectively).

- (a) Prove that every team game (see Problem 5) is isomorphic to a congestion game.
- (b) Prove that every dummy game (see Problem 5) is isomorphic to a congestion game.
- (c) Prove that every potential game is isomorphic to a congestion game.

- Consider n identical machines and m selfish jobs (the players). Each job j has a processing time p_j. Once jobs have chosen machines, the jobs on each machine are processed serially from shortest to longest. (You can assume that the p_j's are distinct.) For example, if jobs with processing times $1, 3$, and 5 are scheduled on a common machine, then they will complete at times $1, 4$, and 9, respectively. The following questions concern the game in which players choose machines in order to minimize their completion times. The objective function as a planner is to minimize the total completion time $\sum_{j=1}^m C_j$, where C_j is the completion time job j.

  - (a) Define the rank R_j of job j in a schedule as the number of jobs on j's machine with processing time at least p_j (including j itself). For example, if jobs with processing times $1, 3$, and 5 are scheduled on a common machine, then they have ranks $3, 2$, and 1, respectively.

    Prove that in these scheduling games, the objective function value of an outcome can also be written as $\sum_{j=1}^m p_j R_j$.
  - (b) Prove that the following algorithm produces an optimal outcome: (i) sort the jobs from largest to smallest; (ii) for $i = 1, 2, \ldots, m$, assign the i^{th} job in this ordering to machine i mod n (where machine 0 means machine n).
  - (c) Prove that for every such scheduling game, the expected objective function value of every coarse correlated equilibrium is at most twice that of an optimal outcome.

    (Hint: The (λ, μ)-smoothness condition (see notes provided) was required for all pairs $\boldsymbol{s}^*, \boldsymbol{s}$ of outcomes. Weaken the definition so that this condition only needs to hold for some optimal outcome $\boldsymbol{s}^*$ and all outcomes $\boldsymbol{s}$. Observe that PoA of coarse correlated equilibria remains at most $\frac{\lambda}{1-\mu}$ assuming only this weaker condition (with the same proof as before). Prove that this scheduling game satisfies this weaker condition for $\lambda = 2$ and $\mu = 0$.)