CS 598RM: Algorithmic Game Theory, Spring 2017
HW 1 (due on Tuesday, Feb 14th at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 30 points.
2. Feel free to *discuss* with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Read the instructions regarding submission details and grading on the webpage. In particular,
   - Type-written submissions are preferred (Using LaTeX/Word as convenient), but hand-written are acceptable - only if your handwriting is extremely pretty to look at. Also, scan it *properly* (use a scanner or a scanning app such as CamScanner), and upload a PDF file on compass2g.
   - No late assignments will be accepted.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available - link can be found on the course webpage.

---

1. (5 points) Consider a two player game where pure-action sets of player one and two are respectively S_1 and S_2. Let $m = |S_1|$ and $n = |S_2|$. Then such a game can be represented by two $m \times n$ dimensional matrices (A, B). Here, player 1 plays $i \in S_1$ and 2 plays $j \in S_2$ then their respective payoffs are $A(i,j)$ and $B(i,j)$. Let Δ_1 and Δ_2 be the set of probability distributions over S_1 and S_2 respectively (sets of mixed-strategies of the players). Consider the following function $f : \Delta_1 \times \Delta_2 \rightarrow \Delta_1 \times \Delta_2$ defined by Nash, where $(x', y') = f(x, y)$:

$$\forall i \in S_1 : \quad x'_i = \frac{x_i + \sigma_i(x,y)}{\sum_{k \in S_1} x_k + \sigma_k(x,y)} \quad \text{where} \quad \sigma_i(x,y) = \max\{0, (Ay)_i - x^T A y\}$$
$$\forall j \in S_2 : \quad y'_j = \frac{y_j + \tau_j(x,y)}{\sum_{k \in S_2} y_k + \tau_k(x,y)} \quad \text{where} \quad \tau_j(x,y) = \max\{0, (x^T B)_j - x^T B y\}$$

   Show that if (x, y) is a fixed-point of f, i.e., $x' = x$ and $y' = y$, then $\forall i \in S_1$, $\sigma_i(x,y) = 0$ and $\forall j \in S_2, \tau_j(x,y) = 0$, and in turn (x, y) is a Nash equilibrium of game (A, B).

2. (5 points) Given the two player game (A, B) of Problem 1 where $A(i,j), B(i,j) > 0, \ \forall i \in S_1, \forall j \in S_2$, consider the symmetric game (C, C^T) with the following $(m+n) \times (m+n)$-dimensional block-matrix:

$$C = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix}$$

Show that a symmetric Nash equilibrium of game (C, C^T) gives a Nash equilibrium of game (A, B). (That is, show that a symmetric equilibrium (z, z) of the symmetric game, can be used to easily obtain an equilibrium (x, y) of the game (A, B).)

3. (5 points) Show that if a mixed-strategy profile (x, y) is a Nash equilibrium of game (A, B), then matrix P where $P_{ij} = x_i * y_j$ is a correlated equilibrium (CE) of the game.

4. There is a weaker notion than CE called coarse-correlated equilibrium (CCE). Here, the mediator announces the joint distribution matrix P, and asks each player to opt in or out before suggesting them any actions. If a player chooses to opt out, then it can play whatever it wants; on the other hand, if it chooses to opt in, then it has to play what the mediator suggests. In other words, unlike CE, a player can not get the suggestions and then choose to not play what is suggested. Matrix P is called CCE of a game (A, B) if no player wants to opt out IF everyone else is opting in.

   (a) (5 points) Show that every correlated equilibrium is a coarse-correlated equilibrium.

   (b) (5 points) Show that all the coarse-correlated equilibria of game (A, B) can be captured by a linear feasibility problem formulation.

5. (5 points) Problem 1.2 of the AGT book.

The remaining problems are for self study. Do *NOT* submit for grading.

- Problem 1.3 of the AGT book.
- Show that checking if a two-player game has more than one Nash equilibrium is NP-hard (hint: reduce from the problem of checking if a graph has clique of size k.)
- (Open problem) Rank of a two-player game (A, B) as $rank(A + B)$. Adsul et al (STOC'11) showed that computing Nash equilibrium of a rank-1 game can be done in polynomial time by reducing the problem to 1-dimensional fixed-point. On the other hand games with rank 2 and more are PPAD-hard.

  Show that checking if a rank-1 game has more than one Nash equilibrium is NP-hard. NP-hardness for even constant rank would be good.
- Problem 1.5 of the AGT book.

- 1-dimensional Sperner's is defined on a 1-dimensional grid from $[0, 2^n - 1]$, with each integer being a grid point. There are two colors red and blue represented by 0 and 1 respectively. There is a Boolean circuit named *Color* which outputs color (0/1 bit) of a grid point given its bit representation, such that, Color(0)=red, Color($2^n - 1$)=blue, and the rest gets any color. Show that there exists an integer $0 \leq k \leq 2^n - 1$ such that Color(k)=red and Color($k+1$)=blue. Furthermore, we can compute it in $O(n)$ calls to the Boolean circuit "Color". Finally, show that checking if there are more than one such ks is NP-hard (hint: reduce from 3-SAT).