NE Computation

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(Alice, Bob)

Recall : 2-player games.

Both players have n strategies/moves.

Representation : $A, B \in \mathbb{R}^{n \times n}$

Alice plays $i \in [n] = \{1, \ldots, n\}$
Bob " $j \in [n]$

payoffs are A_{ij} & B_{ij} respectively.

Set of randomized strategies $= \Delta_n = \left\{ \bar{z} \in [0,1]^n \,\middle|\, \sum_{i=1}^{n} z_i = 1 \right\}$

Alice plays $x \in \Delta_n$
Bob " $y \in \Delta_n$

Payoffs are $x^T A y$ for Alice
$x^T B y$ for Bob.

* Nash Equilibrium : No unilateral deviation is beneficial.

(x, y) is NE iff
For Alice: $x^T A y \geq z^T A y$, $\forall z \in \Delta_n$
For Bob: $x^T B y \geq x^T B z$, $\forall z \in \Delta_n$

↓ characterization.

$$\boxed{\begin{aligned} &\forall i \in [n], \\ &x_i > 0 \Rightarrow (Ay)_i = \max_k (Ay)_k \rightarrow \pi_A \\ &y_j > 0 \Rightarrow (x^T B)_j = \max_k (x^T B)_k \rightarrow \pi_B \end{aligned}}$$

↓

max : $x^T (A+B) y - \pi_A - \pi_B$

$$\max: x^T(A+B)y - \pi_A - \pi_B$$

$$(Ay)_i \le \pi_A, \quad \forall i \in [n]$$

$$(x^TB)_j \le \pi_B, \quad \forall j \in [n]$$

$$x, y \in \Delta_n$$

If $B = -A$ then ↑ is an LP.

★ Algorithm I: Enumerative.

Observation: (x^*, y^*) NE of (A, B)

$$S_A = Supp(x^*) = \{ i \in [n] \mid x_i^* > 0 \}$$

$$S_B = Supp(y^*) = \{ j \in [n] \mid y_j^* > 0 \}$$

Linear Feasibility Formulation

$LF(S_A, S_B)$

$$\boxed{\begin{array}{lll} (Ay)_i \le \pi_A & \forall i \in [n] & \to 1 \\ (x^TB)_j \le \pi_B & \forall j \in [n] & \to 2 \\ x, y \in \Delta_n & & \\ (Ay)_i = \pi_A, & \forall i \in S_A & \to 3 \\ x_i = 0, & \forall i \notin S_A & \to 4 \\ (x^TB)_j = \pi_A, & \forall j \in S_B & \to 5 \\ y_j = 0, & \forall j \notin S_B & \to 6 \end{array}}$$

$$\pi_A = \max_K (Ay^*)_K \qquad \pi_B = \max_K (x^{*T}B)_K$$

__Claim:__ $(x^*, y^*, \pi_A, \pi_B) \in LF(S_A, S_B)$

Lemma: $S_A, S_B \subseteq [n]$, $S_A, S_B \neq \emptyset$. if $LF(S_A, S_B)$ then

Lemma: $S_A, S_B \subseteq [n]$, $S_A, S_B \neq \phi$. If $\exists (x, y, \pi_A, \pi_B) \in LF(S_A, S_B)$ then (x, y) is a NE.

Pf: $\max_k (Ay)_k = \pi_A \to (*)$ $(\because (1))$

$\forall i \in [n]$, $x_i > 0 \Rightarrow i \in S_A \Rightarrow (Ay)_i = \pi_A$
$(\because (4))$ $(\because (3))$

$(\because *) \Rightarrow (Ay)_i \geq \max_{k \in [n]} (Ay)_k$

$\Rightarrow (Ay)_i = \max_{k \in [n]} (Ay)_k$

Algorithm: For every $S_A \subseteq [n]$, $S_B \subseteq [n]$, $S_A, S_B \neq 0$

check if $LF(S_A, S_B) \neq \emptyset$.

$\to$ then o/p $(x, y, \pi_A, \pi_B) \in LF(S_A, S_B)$

Running time: $2^n \times 2^n\, poly(n) \longrightarrow 2^n \times poly(n)$

NE computation in 2-player games is PPAD-hard.

$\frac{1}{poly(n)}$-NE " " "

✓ $O(1)$-NE " $n^{O(\log n)}$ Quasi-poly.

ϵ-NE: (x, y) s.t. $A, B \in [0,1]^{n \times n}$

$x^T A y \geq z^T A y \boxed{- \epsilon}$, $\forall z \in \Delta_n$

ϵ - NE.

$$10^{20}\left(x^T A y \geq z^T A y \boxed{-\epsilon}\right), \quad \forall z \in \Delta_n$$
$$x^T B y \geq x^T B z \boxed{-\epsilon}, \quad \forall z \in \Delta_n$$

(x, y) is NE (A, B)

$\rightarrow (x, y)$ is NE $(\alpha A, \quad \beta B)$ $\quad \alpha, \beta \geq 0$

(x, y) is NE $(\alpha A + \gamma, \quad \beta B + \delta)$ $\quad \gamma, \delta \in \mathbb{R}$

$\epsilon = O(1)$

Aim for running-time $n^{O\left(\frac{\log n}{\epsilon^2}\right)} = n^{O(\log n)}$

[Rubinstein'16] This is the best assuming ETH
for PPAD

ETH $\downarrow$ Exponential Time Hypothesis

Algorithm 2: [Lipton-Markakis-Mehta '03]

(A, B) $\qquad C = A + B$

QP:

$$\max: x^T C y - \pi_A - \pi_B$$
$$(Ay)_i \leq \pi_A \quad \forall i \in [n]$$
$$(x^T B)_j \leq \pi_B \quad \forall j \in [n]$$
$$x, y \in \Delta_n$$

Idea: Guess vector (Cy) at some NE

Idea: Guess vector (Cy) at some NE say δ, replace in Q.P.

$$
\text{LP}(\delta) \quad \left(\tfrac{1}{\epsilon}\right)^n \quad
\boxed{
\begin{aligned}
\max:\ & x^T.\delta - \pi_1 - \pi_2 \\
& \left\{
\begin{aligned}
(Ay)_i &\le \pi_A & \forall i \in [n] \rightarrow ① \\
(x^T B)_j &\le \pi_B & \forall j \in [n] \rightarrow ②
\end{aligned}
\right. \\
& \delta_i \le (Cy)_i \qquad \forall i \in [n] \rightarrow ③ \\
& x, y \in \Delta_n
\end{aligned}
}
$$

Lemma 1: $\delta \in \mathbb{R}^n$, OPT value $(\delta) \le 0$

Pf: (x, y, π_A, π_B) feasible in $\text{LP}(\delta)$ then

$x^T A y \le \pi_A$ ($\because$ ①) $\quad x^T B y \le \pi_B$ ($\because$ ②)

$$\Rightarrow \quad 0 \ge x^T(A+B)y - \pi_A - \pi_B = x^T(Cy) - \pi_A - \pi_B$$
$$\ge x^T.\delta - \pi_A - \pi_B \quad (\because ③)$$

$\square$

Lemma 2: Suppose (x^*, y^*) is a NE.

& δ^* is s.t. $\delta_i^* \le (Cy^*)_i \quad \forall i \in [n]$

$$|\delta_i^* - (Cy^*)_i| < \epsilon \quad \forall i$$

Then, OPT value $(\delta^*) \ge -\epsilon$

Pf:

$\pi_A^* = \max_k (Ay^*)_k$, $\quad \pi_B^* = \max_k (x^{*T}B)_k \Rightarrow x^{*T}Ay^* = \pi_A^*,\ x^{*T}By^* = \pi_B^*$

$(x^*, y^*, \pi_A^*, \pi_B^*)$ is feasible in $\text{LP}(\delta^*)$

$(x^*, y^*, \pi_A^*, \pi_B^*)$ is feasible in LP(δ)

$\text{cost}(\ ) = x^{*T}\cdot\delta^* - \pi_A^* - \pi_B^* \geq x^{*T}(cy^* - \epsilon) - \pi_A^* - \pi_B^*$

$= x^* C y^* - \epsilon - x^{*T}Ay^* - x^{*T}By^*$ ($\because (x^*, y^*)$ is NE)

$= \cancel{x^* C y^*} - \epsilon - \cancel{x^* C y^*}$ ($\because A+B=C$)

$= -\epsilon$ ∎

Lemma 3: If optValue(δ) $\geq -\epsilon$ $\leftarrow$

$(x, y, \pi_A, \pi_B) \in$ OPT of LP(δ), (x,y) is a NE.

Pf:

① $\Rightarrow (\pi_A - \epsilon) \leq x^TAy \leq \pi_A \Rightarrow x^TAy - \pi_A \leq 0$

② $\Rightarrow \underbrace{(\pi_B - \epsilon) \leq}_{\text{TPT.}} x^TBy \leq \pi_B \Rightarrow x^TBy - \pi_B \leq 0$

$\Big\} \rightarrow$ (★)

$-\epsilon \leq x^T\cdot\delta - \pi_A - \pi_B \leq x^T(Cy) - \pi_A - \pi_B$ ($\because$ ③)

$= (x^TAy - \pi_A) + (x^TBy - \pi_B) \rightarrow$ (★★)

($\because C = A+B$)

(★), (★★) $\Rightarrow x^TAy - \pi_A \geq -\epsilon,\quad x^TBy - \pi_B \geq -\epsilon$ ∎

$\Rightarrow (x,y)$ is NE.

—— x —————— x ——————

... to guess $\delta^* \sim (cy^*)$ for (x^*, y^*) NE.

$\hat{\epsilon} \sim (C\underline{y})$: NE.

★ How to guess

$y \sim y^*$ Sample strategy $j \in [n]$ w.p. y^*_j

S = set of K sampled strategies as per y^*.

S is a multiset of strategies.

y = uniform distribution over S.

$$y_j = \frac{\#\text{ times } j \text{ appears in } S}{K}.$$

★ Focus on $i \in [n]$: $|(Cy)_i - (Cy^*)_i| \leq ?$ with what probability?

$C \in [0,1]$

w.r.t. $(Cy^*)_i$

$X_1, \ldots, X_K$

X_ℓ takes value C_{ij} w.p. y^*_j

$E[X_\ell] = (Cy^*)_i$

$\bar{X} = \frac{1}{K}\left[X_1 + \ldots + X_K\right] = (Cy)_i$

$E[\bar{X}] = (Cy^*)_i$

Hoeffding's ineq:

$| \quad | \geq 2e^{-\frac{K^2\epsilon^2}{K}} = 2e^{-K\epsilon^2}$

Hoeffding's ineq.

$$\Pr\left[\left|\bar{X} - E[\bar{X}]\right| > \epsilon\right] < 2e^{-\frac{n\epsilon^2}{K}} = 2e$$

$\bar{X} = (Cy)_i \qquad E[\bar{X}] = (Cy^*)_i$

$\star$ $\forall i \in [n]$, $|(Cy)_i - (Cy^*)_i| < \epsilon$

Good-event

Bad-event: $\exists i \in [n]$, $|(Cy)_i - (Cy^*)_i| > \epsilon$

$$\Pr[\text{Bad-event}] \le 2n\, e^{-K\epsilon^2} < 1$$

$$\Leftrightarrow \quad 2n < e^{K\epsilon^2} \Leftrightarrow K > \frac{\log 2n + 1}{\epsilon^2}$$

$\Downarrow$

$$\Pr[\text{Good-event}] > 0$$

$\Rightarrow$ $\exists y$ that is uniform dist. over a multiset S, $|S| = K = O\left(\frac{\log n}{\epsilon^2}\right)$

s.t. $\delta = Cy$ we have $|\delta_i - (Cy^*)_i| < \epsilon$ $\forall i$

$$S = \{\underset{1..n}{\overset{1}{\square}} \quad \underset{1..n}{\overset{2}{\square}} \quad \cdots \quad \underset{1..n}{\overset{\frac{\log n}{\epsilon^2}}{\square}}\}$$

$n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ many possible multisets of size $O\left(\frac{\log n}{\epsilon^2}\right)$

multisets of size $O\left(\frac{\log n}{\epsilon^2}\right)$

Algorithm:

For each multiset S, $|S| \leq \frac{2 \log n}{\epsilon^2}$

y = uniform dist over S

$\delta = Cy$

$(x, y, \pi_1, \pi_2) \in LP(\delta)$

if optValue $(LP(\delta)) \geq -\epsilon$ then

o/p (x, y)

C is low-rank.

$\text{rank}(C) = r$ can you get $r^{O(r/\epsilon)} \text{poly}(n)$

$r = O(1)$

$$C = \begin{bmatrix} & \\ & \\ & \end{bmatrix}_{n \times r} \begin{bmatrix} & & \end{bmatrix}_{r \times n}$$

$[0, 1]$