Revenue Optimal Auctions

Thursday, November 12, 2020 1:57 PM

Last lec:

① DSIC

② S.W. maximization → Good governments

↓

② Revenue maximization. → Goal of a private company

③ poly-time computation.

Example: single item, single bidder. $v \sim U[0,1]$

→ Rev Vickery $= 0$

→ Post take-it-or-leave-it price p.

DSIC Indirect Mechanism.

$$\max \text{Rev} = \max_p p \cdot \Pr_{v \sim U[0,1]}[v \geq p]$$

$$= \max_p p(1-p)$$

$$= \frac{1}{2}\left(1-\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$$\frac{d}{dp}(p(1-p)) = 0$$
$$1-2p = 0 \Rightarrow p = \frac{1}{2}$$

☆ Two bidders $v_1, v_2 \sim U[0,1]$

→ $\mathbb{E}\left[\begin{matrix}\text{Rev vickery}\\ \text{Auction}\end{matrix}\right] = \mathbb{E}\left[\begin{matrix}\text{second highest}\\ \text{value in } \{v_1, v_2\}\end{matrix}\right] = \frac{1}{3}$

→ Highest bidder wins if the bid $\geq \frac{1}{2}$

Pays $\max\left\{\text{second highest bid}, \frac{1}{2}\right\}$

if $v_1, v_2 < \frac{1}{2}$ then no one gets the item

if $v_1 \geq \frac{1}{2} > v_2$ then 1 wins pay $\frac{1}{2}$

if $v_1 > v_2 \geq \frac{1}{2}$ then 1 wins pays v_2.

$$\mathbb{E}[\text{Rev}] = \frac{5}{12} > \frac{1}{3}$$

Goal: What is the maximum Revenue we can achieve with a DSIC mechanism?

Goal: What is the max ... achieve with a DSIC mechanism?

☆ General Single-parameter setting:

$\{1,\dots,n\}$, $v_i \sim F_i$, $F = \prod_{i=1}^{n} F_i$, set of feasible allocations: X

v_i: Private, F_i: Public

f_i: density func.
F_i: cumulative density func.

Want DSIC, Revenue maximizing mechanism. (x, p), $x \in X$

Myerson's Lemma: DSIC $\iff$ $\boxed{x_i \text{ is monotone } \forall i}$, & p is as per Myerson's formula

design space.

Design allocation rule x s.t. accordingly computed p maximizes the rev.

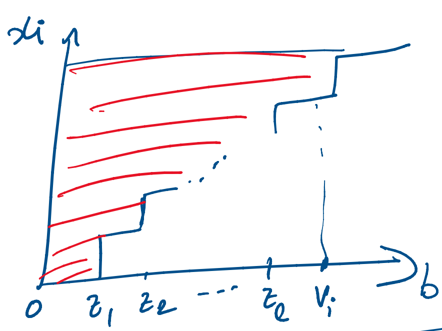
$$\max \sum_{i=1}^{n} \mathop{E}_{v \sim F}\left[P_i(v)\right]$$

$$= \sum_{i=1}^{n} \mathop{E}_{v_{-i} \sim F_{-i}} \underbrace{\mathop{E}_{v_i \sim F_i} P_i(v_i, v_{-i})}_{\text{Focus}} \rightarrow (1)$$

fix

Recall Myerson's payment formula: $\sum_{d=1}^{l} z_d \, \Delta x_i(z_d, v_{-i})$

$$P_i(v_i, v_{-i}) = \underbrace{\int_0^{v_i} z \, x_i'(z, v_{-i}) \, dz}$$

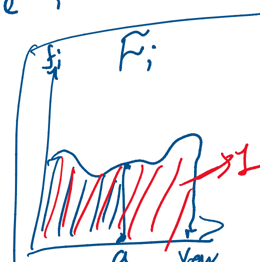


Replace in (1)

Fix i, Fix v_{-i}

$$\mathop{E}_{v_i \sim F_i}\left[P_i(v_i, v_{-i})\right] = \int_0^{v_{max}} \boxed{P_i(v_i, v_{-i})} \, f_i(v_i) \, dv_i$$

$$= \int_0^{v_{max}} \left(\int_0^{v_i} z \, x_i'(z, v_{-i}) \, dz \right) f_i(v_i) \, dv_i$$



$f_i(a)$: "prob" that $v_i = a$

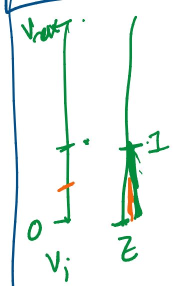
$F_i(a)$ = Prob that $v_i \le a$

$$= \int_0 \left(\int_0 z\, x_i'(z, v_{-i})\, dz \right) \ldots$$

$$= \int_0^{V_{max}} \left(\int_z^{V_{max}} \underline{f_i(v_i)}\, dv_i \right) z \cdot x_i'(z, v_{-i})\, dz$$

$$= \int_0^{V_{max}} \underbrace{\left(1 - F_i(z)\right) \cdot z}_{f} \cdot \underbrace{x_i'(z, v_{-i})\, dz}_{g'}$$

$F_i(a)$ = Prob that $v_i \leq a$



$\int (f g)' = \int f' g + \int f g'$

$\int f \cdot g' = f \cdot g - \int f' g$

Integration by parts

$$= \left. \left(1 - F_i(z)\right) \cdot z \cdot x_i(z, v_{-i}) \right|_0^{V_{max}}$$

$$0 =$$

$$- \int_0^{V_{max}} \left(1 - F_i(z) - z \cdot f_i(z)\right) \cdot x_i(z, v_{-i})\, dz$$

Renaming change $z \to v_i$

$$= \int_0^{V_{max}} \left(\frac{v_i f_i(v_i) - 1 + F_i(v_i)}{f(v_i)} \right) x_i(v_i, v_{-i})\, dv_i \cdot f_i(v_i)$$

Interpretation next.

$$= \int_0^{V_{max}} \underbrace{\left(v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right)}_{\text{"Virtual Value"}} \underline{x_i(v_i, v_{-i})}\, \underline{f_i(v_i)}\, dv_i$$

$\phi_i(v_i) =$

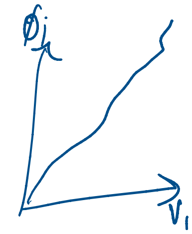
$$= \mathop{\mathbb{E}}_{v_i \sim F_i} \Big[\underbrace{\phi_i(v_i) \cdot x_i(v_i, v_{-i})}_{\text{Virtual welfare of agent } i} \Big] = \mathop{\mathbb{E}}_{v_i \sim F_i} \left[p_i(v_i, v_{-i}) \right]$$

$$\max \mathbb{E}\left[\text{Rev}\right] = \max \mathbb{E}_v \left[\sum_i p_i(v) \right]$$

$$\max E[\text{Rev}] = \max \underset{v\sim F}{\mathbb{E}}\left[\sum_i \ldots\right]$$

$$= \max \underset{v\sim F}{\mathbb{E}}\left[\underbrace{\sum_i \phi_i(v_i)\, x_i(v_i, v_{-i})}_{\text{virtual social welfare.}}\right]$$

$$\boxed{x^*(v) = \underset{x\in\mathcal{X}}{\arg\max} \sum_i \phi_i(v_i)\, x_i(v_i, v_{-i})}$$



Myerson's formula gives p^*

DSIC $\iff$ x^* is monotone $\overset{\text{Exe}}{\Longleftarrow}$ ϕ_i's are monotone functions $\forall i$

$\Updownarrow$

F_i is "Regular".

Example: Single item, n-bidders.

$\forall i$: $v_i \sim F_i$ $\quad \mathcal{X} = \left\{ x\in\{0,1\}^n \;\middle|\; \sum_{i=1}^n x_i \le 1 \right\}$

assume $b=v$ ($\because$ DSIC)

$$x^*(v) = \underset{x\in\mathcal{X}}{\arg\max} \sum_{i=1}^n \phi_i(v_i)\, x_i = \max\left\{ \max_i \phi_i(v_i),\ 0 \right\}$$

$\boxed{b_i = v_i \ \forall i \ \because \text{DSIC.}}$

$i^* = \underset{i}{\arg\max}\ \phi_i(v_i)$ and

if $\phi_{i^*}(v_{i^*}) \ge 0$ then i^* gets the item

... gets the item.

∵ DSIC

if $\phi_{i^*}(v_{i^*}) \geq 0$

else no one gets the item.

$F_i = F$ Regular $\forall i$

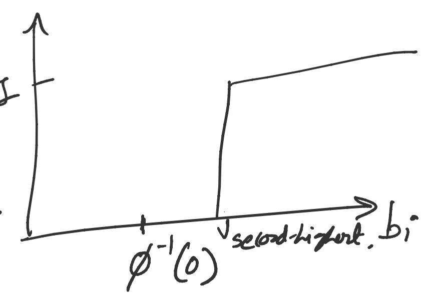
$\phi_i = \phi$ is monotone.

i^* gets the item because

$\phi(v_{i^*}) \geq 0$, & $\phi(v_{i^*}) \geq \phi(v_j)$ $\forall j \neq i^*$

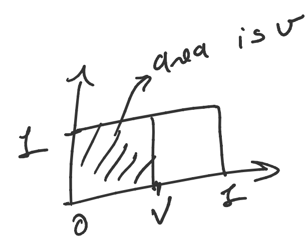
$\Updownarrow \qquad\qquad\qquad \Updownarrow$

$v_{i^*} \geq \phi^{-1}(0) \qquad v_{i^*} \geq v_j, \forall j$



$$P_{i^*} = \max\left\{ \phi^{-1}(0) , \text{ second-highest bid} \right\}$$

Reserve price



$F = U[0,1]$

$$\phi(v) = v - \frac{1 - F(v)}{f(v)} = 0$$

$$= v - \frac{(1-v)}{1} = 0 \Rightarrow 2v - 1 = 0$$

$$\Rightarrow v = \frac{1}{2}$$

$\phi^{-1}(0) = \frac{1}{2}$ = reserve price.

$\forall i: v_i \sim F_i$ different but regular

ϕ_i's are monotone but different.

$b_i = v_i$
∵ DSIC

$$i^* = \underset{i}{\arg\max}\ \phi_i(v_i)$$

i^* if $\phi_{i^*}(v_{i^*}) \geq 0$

give to i^* if $\phi_{i^*}(v_{i^*}) \geq 0$

otherwise give to none.

$$\phi_{i^*}(v_{i^*}) \geq \phi_i(v_i) \quad \forall i \neq i^*$$

$$\not\Rightarrow \quad v_{i^*} \geq v_i$$

(e.g. $F_1 = U[0, 1]$, $F_2 = U[0, \frac{1}{2}]$

$$\phi_1(v_1) \geq \dots \geq \phi_n(v_n)$$

1 wins because $\phi_1(v_1) \geq 0 \Rightarrow v_1 \geq \phi_1^{-1}(0)$

& $\phi_1(v_1) \geq \phi_2(v_2) \Rightarrow v_1 \geq \underline{\phi_1^{-1}(\phi_2(v_2))}$

$$\text{Pay}_1 = \max\{\phi_1^{-1}(0), \phi_1^{-1}(\phi_2(v_2))\}$$

$\phi_1^{-1}(0)$: agent 1's reserve price.

$\phi_1^{-1}(\phi_2(v_2))$: "second-highest bid"

Unknown distribution

(Prior Independent Auctions)

Bulow-Klemperer (1996)

i.i.d $F_i = F$ regular $\rightarrow$ ϕ is con. virtual value func.

Thm: $\mathbb{E}\begin{bmatrix}\text{Rev of} \\ \text{Vickery} \\ \text{w/ } (n+1) \text{ bidders}\end{bmatrix} \geq \mathbb{E}\begin{bmatrix}\text{Rev of} \\ \text{OPT auction} \\ \text{w/ } n \text{ bidders}\end{bmatrix}$

pf: Suppose we "HAVE TO" sell the item.

$$X = \{x \in \{0,1\}^n \mid \sum_{i=1}^{n} x_i = 1\}.$$

Pf: Suppose we ...

$$\mathcal{X} = \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} x_i = 1 \right\}.$$

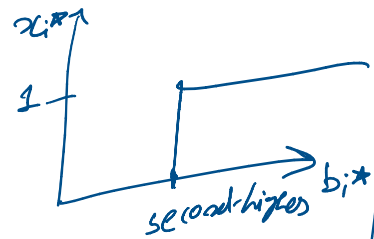
OPT that always allocates the item

$$x^* = \underset{\substack{x \in \mathcal{X} \\ \equiv \\ x \geq 0, \\ \sum_{i=1}^n x_i = 1}}{\arg\max} \sum_i \phi(v_i)\, x_i$$

$$i^* = \arg\max_i \phi_i(v_i) = \arg\max_i v_i$$

x^*: will give the item to highest-bidder, say i^*

P_{i^*} = second-highest bid.



Vickrey Auction.

Auction A w/ (n+1) bidders.

always allocates the item

Auction w/ n bidders

Step 1: Runs OPT auction w/ n bidders

Step 2: if item is still unallocated then give it for free to the $(n+1)^{th}$ bidder.

$$\mathbb{E}\left[\begin{matrix}\text{Rev of} \\ \text{Vickrey} \equiv \left(\begin{matrix}\text{OPT when} \\ \text{have to allocate} \\ \text{the item}\end{matrix}\right) \\ \text{w/ } (n+1) \text{ bidder}\end{matrix}\right] \geq \mathbb{E}\left[\begin{matrix}\text{Rev of} \\ \text{Auction } A\end{matrix}\right] = \mathbb{E}\left[\begin{matrix}\text{OPT Rev} \\ \text{w/ } n \text{ bidders}\end{matrix}\right]$$