

Last Lec: Awesome Auctions.  $(x, p)$  <sup>allocation</sup> <sup>payment</sup> (Single Parameter)

① DSIC (truthfulness)

② s.w. maximization

③ Polynomial-time computable

$x_i(b_i)$  is monotone  $\forall i$

Step 1: Decides allocation  $x$  that maximizes s.w. (assuming  $b_i = v_i$ )

Step 2: Apply Myerson's payment formula to decide payments  $P$ .

Q: s.w. maximizing allocation is monotone? YES!

$(b_1, \dots, b_m)$  bids of agents.  $X$ : feasible allocation set (assuming  $b_i = v_i$ )

$$\text{Algorithmic problem: } \max_{x \in X} \sum_{i=1}^n b_i x_i$$

$b = (b_1, \dots, b_i, \dots, b_m) \rightarrow (x_1, \dots, x_m)$

$x'_i \geq x_i?$

$b' = (b_1, \dots, b'_i, \dots, b_m) \rightarrow (x'_1, \dots, x'_m)$

YES!

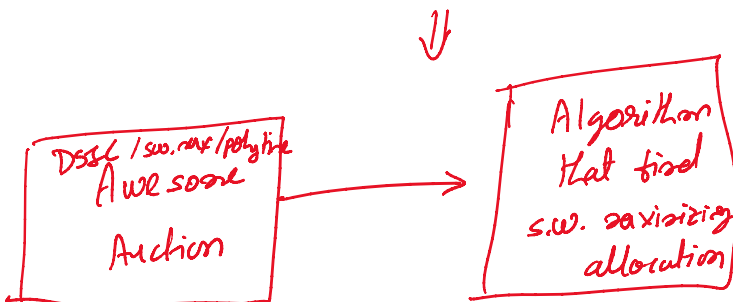
$b'_k = b_k \quad \forall k \neq i$

$$\sum_{i=1}^n b_i x_i \geq \sum_{i=1}^n b_i x'_i$$

$$\sum b'_i x'_i \geq \sum b_i x_i$$

$$\Rightarrow (b'_i - b_i) \underbrace{(x'_i - x_i)}_{\geq 0} \geq 0$$

$\underbrace{\quad}_{\geq 0} \Rightarrow \underbrace{\quad}_{\geq 0}$



... k-item, sponsored search.

Examples: 1-item, k-item, sponsored search.  
 Vickrey

TV Ad. slot Auctions.:

$W$  secs of ad window available.

$n$  companies/agents want to show their ads.

$i$ th agent's ad is of  $w_i$  secs.  $\rightarrow$  public knowledge

" " value to show the ad is  $v_i \rightarrow$  private knowledge.

$$X = \left\{ x \in \{0,1\}^n \mid \sum_{i=1}^n w_i x_i \leq W \right\}$$

$(b_1, \dots, b_n)$  are the bids of agents.

allocation  $x^* \in \arg \max_{x \in X} \sum_{i=1}^n b_i x_i$

(can this be computed in poly-time?)

NO!

Knapsack Problem NP-hard.

Greedy Approach:  $\frac{b_1}{w_1} > \frac{b_2}{w_2} > \dots > \frac{b_n}{w_n}$

Step 1: allocate in this order until possible  
 suppose we allocate  $1, \dots, d$   
 $\sum_{i=1}^d w_i x_i \leq W$  &  $\sum_{i=1}^{d+1} w_i x_i > W$

Step 2:  $b_{max} =$  highest bid.  
 $\hookrightarrow i_{max}$

if  $b_{max} > \sum_{i=1}^d b_i x_i$  then allocate only  $i_{max}$

else allocate  $\{1, \dots, d\}$

Ex:  $\frac{0.2}{0.01} > \frac{0.1}{0.01} > \frac{5}{1}$   $W=1$

" " " " 2-approximation to the opt.

Else allocate  $\{1, \dots, n\}$

Thm: Greedy allocation is 2-approximation to the opt.  

$$\left( \frac{\text{opt s.w.}}{2} \leq \frac{\text{s.w. of greedy approach}}{\text{Greedy.}} \right)$$

PS: Int opt  $\leq$  Fractional opt.  

$$= \sum_{i=1}^d b_i x_i + b_{d+1} \frac{(W - \sum_{i=1}^d w_i)}{1} \leq 1$$
  

$$\leq \text{Greedy} + b_{d+1} \leq \text{Greedy} + b_{\max} \leq 2 \text{ Greedy.}$$

$$\text{Greedy} = \max \left\{ \sum_{i=1}^d b_i x_i, b_{\max} \right\}$$

$$\Rightarrow \frac{\text{OPT}}{2} \leq \text{Greedy.}$$

Is Greedy allocation rule monotone? YES.

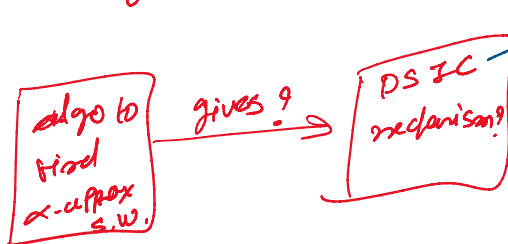
DSIC using Myerson's payment formula.

DSIC  $\frac{1}{2}$ -approx<sup>OPT</sup> s.w., poly-time.

$(1+\epsilon)$ -approx algo. allocation is not monotone!

Lots of work on approximation algo for NP-hard problems.

Q: Can we use these to design DSIC mechanisms with approximately optimal s.w.?



YES, if the allocation rule is monotone.

Is this always the case?

**NO!**

Mostly yes. But not always.  
 e.g.  $X$  is downward close ( $x \in X, y \leq x$  then  $y \in X$ )

e.g.  $x$  is downward close (a...)  
 Then the answer is YES.

BIC: Bayesian Incentive Compatible.  
 (truth telling is a NE.)

General Setting: (eg. selling  $k$ -heterogeneous items)  
 1-item auction

- set of agents  $\{1, \dots, n\}$
- set of outcomes:  $\Omega$
- Every  $i \in \{1, \dots, n\}$  has valuation func.

$$\Omega = \{i\text{-wins} \mid i \in \{1, \dots, n\}\}$$

$$\begin{aligned} V_i(i\text{-wins}) &= v_i \\ V_i(j\text{-wins}) &= 0 \text{ for } j \neq i \end{aligned}$$

$$V_i: \Omega \rightarrow \mathbb{R} \quad (\text{Private})$$

Bids  $b_i: \Omega \rightarrow \mathbb{R}$  to the mechanism.

Mechanism: Decides  $\omega^* \in \Omega$   
 ②  $p_i$  for each  $i \in \{1, \dots, n\}$ .

★ Vickrey-Clark-Groove (VCG): The only DSIC mechanism.

S.w. maximizing outcome:  $\omega^* \in \arg \max_{\omega \in \Omega} \sum_{k=1}^n b_k(\omega)$   
 $v_i(\omega) + \sum_{k \neq i} b_k(\omega)$  if  $b_i = v_i$

$p_i$  = "Externality" (harm)  $i$  causes to the system by participating

$$= \boxed{\max_{\omega \in \Omega} \sum_{k \neq i} b_k(\omega)} - \boxed{\sum_{k \neq i} b_k(\omega^*)}$$

- harm it                      s.w. of others when



$w \in \Omega_{k \neq i}$   
 s.w. of others if  
 i does not participate

...  
 s.w. of others when  
 i participates.

Then: This is DSIC.

PF:

$$U_i = V_i(w^*) - P_i$$

$$= V_i(w^*) + \sum_{k \neq i} b_k(w^*)$$

i wants bid so that  
 this is maximized.

$$- \max_{w \in \Omega_{k \neq i}} \sum_{k \neq i} b_k(w)$$

$h_i(b_i)$   
 agent i cannot  
 influence this term.

i wants  $w^* \in \arg \max_{w \in \Omega} V_i(w) + \sum_{k \neq i} b_k(w)$  is maximized.

This is achieved when  $b_i = V_i$

Issues with VCG:

① Computation issues: like in TV ad auction.

(s.w. maximizing allocation + Myerson's payment rule  $\equiv$  VCG)

② 10 heterogeneous items auction.

to represent  $v_i (\equiv b_i)$  agent needs to specify  $2^{10}$  numbers.

Direct revelation

sol<sup>m</sup>: Indirect mechanisms. (e.g. english/dutch auction)

Revelation Principle:

$\perp$  Had is a  $\rightarrow$  Direct DSIC mechanism.

Revelation mechanism.   
 If there is a DSE in I.M.  $\Rightarrow$  Direct DSIC mechanism.

③ Revenue:

item set  $\{A, B\}$   
 $V_1(A, B) = 1, \quad V_1(A) = V_1(B) = 0$   
 $V_2(A) = V_2(AB) = 1$

$w^*$ :  $1 \leftarrow AB$   
 or  $2 \leftarrow AB$  } s.w. = 1

$P_1 = 1 - 0 = 1$   
 $+ P_2 = 1 - 1 = 0$   


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 Rev = 1

$V_3(B) = V_3(AB) = 1$

$w^*$ :  $3 \leftarrow B$   $V_3(B) = 1$   
 $2 \leftarrow A$   $V_2(A) = 1$   
 $2 = sw$   
 $P_1 = 0$   
 $+ P_2 = 1 - 1 = 0$   
 $+ P_3 = 1 - 1 = 0$   


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 Rev = 0

④ Signaling: (when auction happens in multiple rounds).