# Auction -> Algorithm

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Last Lec: Awesome Auctions. $(\vec{x}, p)$ (allocation, payment) (Single Parameter)

① DSIC (truthfulness) ② s.w. maximization ③ Polynomial-time computable

$\Updownarrow$

$x_i(b_i)$ is monotone $\forall i$

Step 1: Decides allocation $x$ that maximizes s.w. (assuming $b_i = v_i$)

Step 2: Apply Myerson's payment formula to decide payments P.

Q: s.w. maximizing allocation is monotone? YES!

$(b_1, \dots, b_n)$ bids of agents. $X$: feasible allocation set (assuming $b_i = v_i$)

$$\boxed{\underset{x \in X}{\arg\max} \sum_{i=1}^{n} b_i x_i}$$ Algorithmic problem

$b = (b_1, \dots, b_i, \dots, b_n) \rightarrow (x_1, \dots, x_n)$

$\qquad\quad \wedge$

$b' = (b_1, \dots, b_i', \dots, b_n) \rightarrow (x_1', \dots, x_n')$

$\boxed{x_i' \geq x_i ?}$ YES!

$b_k' = b_k \ \forall k \neq i$

$$\left.\begin{aligned} \sum_{i=1}^{n} b_i x_i &\geq \sum_{i=1}^{n} b_i x_i' \\ \sum b_i' x_i' &\geq \sum b_i x_i \end{aligned}\right\} \Rightarrow \underbrace{(b_i' - b_i)}_{\substack{\vee \\ 0}} \underbrace{(x_i' - x_i)}_{\substack{\vee\!\mid \\ 0}} \geq 0 \Rightarrow$$

$\Downarrow$

$$\boxed{\begin{array}{c}\text{DSIC / sw.max / poly time} \\ \text{Awesome Auction}\end{array}} \longrightarrow \boxed{\begin{array}{c}\text{Algorithm} \\ \text{that find} \\ \text{s.w. maximizing} \\ \text{allocation}\end{array}}$$

Single item, k-item, sponsored search.

Examples: 1-item Vickrey, k-item, sponsored search.

## TV Ad. slot Auctions:

$W$ secs of ad window available.

$n$ companies/agents want to show their ads.

$i$th agent's ad is of $w_i$ secs. → public knowledge

" " value to show the ad is $v_i$ → private knowledge.

$$X = \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} w_i x_i \le W \right\}$$

$(b_1, \ldots, b_n)$ are the bids of agents. ($b_i = v_i$ assume)

$$\boxed{\text{allocation} \quad x^* \in \underset{x \in X}{\operatorname{argmax}} \sum_{i=1}^{n} b_i x_i}$$

Can this be computed in poly-time?

NO!

Knapsack problem NP-hard.

**Greedy Approach:**

$$\frac{b_1}{w_1} > \frac{b_2}{w_2} > \cdots > \frac{b_n}{w_n}.$$

Step 1: allocate in this order until possible

suppose we allocate $1, \ldots, d$

$$\sum_{i=1}^{d} w_i x_i \le W \quad \& \quad \sum_{i=1}^{d+1} w_i x_i > W$$

Ex: $\frac{0.2}{0.01} > \frac{0.1}{0.01} > \frac{5}{1}$ $\quad W = 1$

Step 2: $b_{max}$ = highest bid.

↳ $i_{max}$

if $b_{max} > \sum_{i=1}^{d} b_i x_i$ then allocate only $i_{max}$

Else allocate $\{1, \ldots, d\}$

2-approximation to the opt.

Else allocate $\{1, \dots, d\}$

**Thm:** Greedy allocation is 2-approximation to the opt.

$$\left( \frac{\text{opt s.w.}}{2} \le \frac{\text{s.w. of greedy approach}}{\text{Greedy}} \right)$$

**Pf:** 1st opt $\le$ Fractional opt.

$$= \sum_{i=1}^{d} b_i x_i + b_{d+1} \underbrace{\frac{\left(W - \sum_{i=1}^{d} w_i\right)}{w_{d+1}}}_{\le 1}$$

$$\le \text{Greedy} + b_{d+1} \le \text{Greedy} + b_{max}$$

$$\le 2\,\text{Greedy}.$$

$$\boxed{\text{Greedy} = \max\left\{ \sum_{i=1}^{d} b_i x_i ,\ b_{max} \right\}}$$

$$\Rightarrow \frac{\text{OPT}}{2} \le \text{Greedy}. \quad \blacksquare$$

Is Greedy allocation rule monotone? YES.

$\Updownarrow$

DSIC using Myerson's payment formula.

DSIC $\frac{1}{2}$-approx OPT s.w., poly-time. $(1+\epsilon)$-approx algo. allocation is not monotone!

Lots of work on approximation algo for NP-hard problems.

Q: Can we use these to design DSIC mechanisms with approximately optimal s.w?

$\boxed{\text{algo to find } \alpha\text{-approx s.w.}} \xrightarrow{\text{gives?}} \boxed{\text{DSIC mechanism?}} \rightarrow \text{BIC}$

YES, if the allocation rule is monotone.

Is this always the case?

NO!

Mostly yes. But not always.

e.g. $X$ is downward close ($x \in X$, $y \le x$ then $y \in X$)

e.g. $X$ is downward close (

Then the answer is YES.

BIC: Bayesian Incentive Compatible.

(truthtelling is a NE.)

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<u>General Setting</u>: (eg. selling k-heterogeneous items)

→ set of agents $\{1, \dots, n\}$

→ set of outcomes: $\Omega$

→ Every $i \in \{1, \dots, n\}$ has valuation func.

$V_i : \Omega \to \mathbb{R}$ (Private)

Bids $b_i : \Omega \to \mathbb{R}$ to the mechanism.

1-item auction

$\Omega = \{ i\text{-wins} \mid i \in \{1, \dots, n\} \}$

$V_i(i\text{-wins}) = V_i$

$V_i(j\text{-wins}) = 0$ for $j \neq i$

Mechanism: Decides ① $\omega^* \in \Omega$

② $P_i$ for each $i \in \{1, \dots, n\}$.

☆ Vickery-Clark-Groove (VCG): The only DSIC mechanism.

s.w. maximizing outcome: $\omega^* \in \underset{\omega \in \Omega}{\arg\max} \sum_{k=1}^{n} b_k(\omega)$

" $V_i(\omega) + \sum_{k \neq i} b_k(\omega)$ if $b_i = V_i$

$P_i$ = "Externality" (harm) $i$ causes to the system by participating

$$= \boxed{\max_{\omega \in \Omega} \sum_{k \neq i} b_k(\omega)} - \boxed{\sum_{k \neq i} b_k(\omega^*)}$$

s.w. of others when

$- \boxed{\max_{w \in \Omega} \sum_{k \neq i} \ldots}$ ... $\ldots$

s.w. of others if $i$ does not participate

s.w. of others when $i$ participates.

Thm: This is DSIC.

Pf:

$$U_i = V_i(w^*) - P_i$$

$$= \boxed{V_i(w^*) + \sum_{k \neq i} b_k(w^*)} - \underbrace{\max_{w \in \Omega} \sum_{k \neq i} b_k(w)}_{h_i(b_i)}$$

$i$ wants bid so that this is maximized.

agent $i$ cannot influence this term.

$i$ wants $w^* \in \underset{w \in \Omega}{\arg\max}\, V_i(w) + \sum_{k \neq i} b_k(w)$ is maximized.

This is achieved when $b_i = V_i$

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Issues with VCG:

1. Computation issues: like in TV ad auction.

   (s.w. maximizing allocation + Myerson's payment rule $\equiv$ VCG)

2. 10-heterogeneous items auction.

   to represent $v_i$ ($\equiv b_i$) agent needs to specify $2^{10}$ numbers.

   Direct revelation

Sol$^n$: Indirect mechanisms. (e.g. english/dutch auction)

Revealation Principle:

If there is a $\longrightarrow$ Direct DSIC mechanism.

Revelation ...

If there is a DSE in I.M. $\Longrightarrow$ Direct DSIC mechanism.

③ Revenue :

item set $\{A, B\}$

$V_1(A,B) = 1$, $V_1(A) = V_1(B) = 0$

$V_2(A) = V_2(AB) = 1$

$w^*$: $1 \leftarrow AB$ OR $2 \leftarrow AB$ s.w. = 1

$P_1 = 1 - 0 = 1$

$+ P_2 = 1 - 1 = 0$

Rev $= 1$

$\boxed{V_3(B) = V_3(AB) = 1}$

$w^*$: $3 \leftarrow B$, $2 \leftarrow A$

$V_3(B) = 1$

$V_2(A) = 1$

$2 = sw$.

$P_1 = 0$

$+ P_2 = 1 - 1 = 0$

$+ P_3 = 1 - 1 = 0$

Rev $= 0$

④ Signaling : (when auction happens in multiple rounds).