Game Dynamics

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Q_1: How do players "reach" an equilibrium? Depends on the players: what they know, their behavioral model, what they observe ..

Q_2: How fast they reach an equilibrium?

No Good Answers, in general.

For potential Games, things are not too bad!

Simultaneous Best response

	F	T
F	3, 2	0, 0
T	0, 0	2, 3

★ Best Response Dynamics:

Play best response one-player-at-a-time. (Drastic change in strategy)

★ No Regret Dynamics: (ϵ-parameter) Simultaneous change in strategy by players.

→ Multiplicative Weight Update (MWU)

$$x_i^s(t+1) = \left(1 + \epsilon \frac{\text{Payoff}_i^s(t)}{\text{Payoff}_i(t)}\right) \cdot x_i^s(t)$$

★ Best Response Dynamics.

Potential Game $\Rightarrow$ $\phi : S \to \mathbb{R}$. $\left(S = \bigtimes_{i=1}^{K} S_i\right)$

$$c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) = \phi(s_i, s_{-i}) - \phi(s_i', s_{-i}) \quad \forall i,\ \forall s \in S,\ \forall s_i' \in S_i$$

Proposition: For finite potential game, the best response dynamics converges to a pure NE.

Proposition: For finite potential game, the ... converges to a pure NE.

Pf. $\phi_{min} = \min_{s \in S} \phi(s)$ $\qquad s^0 \in S$ start state.

$\phi(s^0) \longrightarrow \phi_{min}$ $\qquad \delta = \min_{\substack{s \in S,\ s_i' \in S_i' \\ \phi(s) \neq \phi(s_i', s_{-i})}} |\phi(s) - \phi(s_i', s_{-i})| > 0$

Suppose current state is s. And player i can improve by changing unilaterally to s_i'

$$\delta \leq c_i(s_i, s_{-i}) - c_i(s_i', s_{-i}) = \phi(s_i, s_{-i}) - \phi(s_i', s_{-i})$$

$$\Rightarrow \phi(s_i', s_{-i}) \leq \phi(s) - \delta$$

$$\# \text{ steps} \leq \frac{\phi(s^0) - \phi_{min}}{\delta}$$

☆ How Fast Does it Converge?

Worst case is exponential (Skopalik & Vocking '08).

Atomic Routing Games.

Thm: Atomic Routing Game with k-players. s.t.

1. All players want to go from the same source to the same destination.
2. Cost functions have "α-bounded jump" property $\forall e \in E,\ \forall i \leq k$ $\quad c_e(i+1) \in [c_e(i), \alpha c_e(i)]$ for $\alpha \geq 1$.
3. Max-gain Best Response: In every round the ... current moves.

3. Max-gain Best Response : In every round the player with max improvement moves.

Then, an ϵ-PNE reached in $O\left(\frac{K\alpha}{\epsilon} \log \frac{\phi(s^0)}{\phi_{min}}\right)$ many steps.

ϵ-PNE : s

$\forall i$: $c_i(s_i', s_{-i}) \geq (1-\epsilon)\, c_i(s) \quad \forall s_i' \in S_i$

Maxgain BRD : If s is not an ϵ-PNE, then the player with max improvement will move.

$\downarrow$

ϵ-PNE

Lemma 1 : Given $s \in S$, $\exists i$, $c_i(s) \geq \frac{\phi(s)}{K}$ $\quad$ (K = # players)

Pf : $\quad f_e$ = # players taking edge e.

$$K \cdot \max_{i=1}^{K} c_i(s) \geq \sum_{i=1}^{K} c_i(s) = \text{Cost}(s) = \sum_{e \in E} f_e \cdot l_e(f_e)$$

$$= \sum_{e \in E} \sum_{i=1}^{f_e} l_e(f_e)$$

$$\geq \sum_{e \in E} \sum_{i=1}^{f_e} l_e(i) \qquad (\because l_e \text{ is nondec.})$$

$$= \phi(s)$$

$$\Rightarrow \max_{i=1}^{K} c_i(s) \geq \frac{\phi(s)}{K}$$

Lemma 2 : At $s \in S$ if player i moves to s_i' as per Maxgain

Lemma 2: At $s \in S$ if player

dynamics then

$$c_i(s) - c_i(s_i', s_{-i}) \geq \frac{\epsilon}{\alpha} c_j(s) \quad \forall j \in \{1, \dots k\}$$

pf:

$$c_i(s_i', s_{-i}) \leq (1-\epsilon)\, c_i(s)$$

$$\Rightarrow \underline{c_i(s) - c_i(s_i', s_{-i}) \geq \epsilon\, c_i(s)} \longrightarrow (*)$$

Fix player $j \in \{1, \dots, k\}$

case I: j also has an ϵ-improving move, say s_j'

$$c_j(s) - c_j(s_j', s_{-j}) \geq \epsilon\, c_j(s) \geq \frac{\epsilon}{\alpha} c_j(s) \quad (\because \alpha \geq 1)$$

$$\wedge$$

$$c_i(s) - c_i(s_i', s_{-i})$$

case II: j does not have ϵ-improving move.

$$c_i(s_i', s_{-i}) \leq (1-\epsilon)\, c_i(s)$$

$$c_j(s_i', s_{-j}) \geq (1-\epsilon)\, c_j(s)$$

k-paths k-paths

$\underline{(s_i', s_{-i})}$, $\underline{(s_i', s_{-j})}$ have $(k-1)$ paths common

$s_1, \dots s_k, s_i'$ are paths from s to t

$$s_{-i} = \{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k\} \cap s_{-j} = \{s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_k\}$$

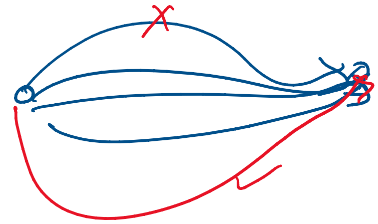
$i < j$

$$= |\{s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_{j-1}, s_{j+1}, \dots, s_k\}| = k-1$$

(s_i', s_{-i}) differ in exactly one path

So, (s_i', s_{-i}) , (s_i', s_{-j}) differ in exactly one path

player i sees $\Downarrow$ f_e^i player j sees $\Downarrow$ f_e^j



$$|f_e^i - f_e^j| \le 1$$

$$\Rightarrow \quad f_e^j \le f_e^i + 1$$

$$(1-\epsilon)\, c_j(s) \le c_j(s_i', s_{-j}) = \sum_{e \in s_i'} c_e(f_e^j)$$

$$\le \sum_{e \in s_i'} c_e(f_e^i + 1)$$

$$\le \sum_{e \in s_i'} \alpha\, c_e(f_e^i)$$

$$= \alpha\, c_i(s_i', s_{-i})$$

$$(1-\epsilon)\, c_i(s) \ge c_i(s_i', s_{-i}) \ge \frac{(1-\epsilon)}{\alpha} c_j(s)$$

$$\Rightarrow \quad \frac{c_j(s)}{\alpha} \le c_i(s)$$

$$(*) \Rightarrow \quad \underline{c_i(s) - c_i(s_i', s_{-i}) \ge \epsilon\, c_i(s) \ge \frac{\epsilon}{\alpha} c_j(s)}$$

<u>Proof of Thm</u> : At time t, the current play s.

i moves to s_i'

Proof

i moves to s_i'

$$\phi(s) - \phi(s_i', s_{-i}) = c_i(s) - c_i(s_i', s_{-i})$$

$$(\because \text{Lemma 2}) \geq \frac{\epsilon}{\alpha} \max_{j=1}^{k} c_j(s)$$

$$(\because \text{Lemma 1}) \geq \frac{\epsilon}{\alpha} \frac{\phi(s)}{k}$$

$$\phi(s_i', s_{-i}) \leq \left(1 - \frac{\epsilon}{\alpha k}\right) \phi(s)$$

$s^0, \ldots\ldots, s^t$

$$\phi_{min} = \phi(s^t) \leq \left(1 - \frac{\epsilon}{\alpha k}\right)^t \phi(s^0)$$

$$\frac{\phi(s^0)}{\phi_{min}} \geq \frac{1}{\left(1 - \frac{\epsilon}{\alpha k}\right)^t}$$

$$\log\left(\frac{\phi(s^0)}{\phi_{min}}\right) \geq -t \log\left(1 - \frac{\epsilon}{\alpha k}\right)$$

$$\geq t\left(\frac{\epsilon}{\alpha k}\right)$$

$$\boxed{t \leq \frac{\alpha k}{\epsilon} \log\left(\frac{\phi(s^0)}{\phi_{min}}\right)}$$

☆ (λ, μ) - Smooth Game.

s^* is the min cost strategy.

$$\text{cost}(s^t) \leq \frac{\lambda}{1-\mu} \text{cost}(s^*) + n$$