Atomic Selfish Routing

Thursday, October 15, 2020 1:57 PM

$G = (V, E)$ directed n/w $\quad \forall e \in E$, c_e is the cost fnc

$c_e : \mathbb{Z}_+ \to \mathbb{R}_+$

non-dec, non-neg.

Players $= \{1, \ldots, n\}$

Player i wants to go from $\overset{\nearrow}{s_i}$ to $\overset{\overset{V}{\uparrow}}{t_i}$ $\quad \in V$

$\boxed{\mathcal{P}_i}$: $s_i \to t_i$ paths , $P_i \in \mathcal{P}_i$

↑ set of pure moves for player i

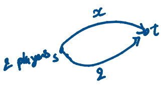
$P = (P_1, \ldots, P_n)$ strategy profile. $\to$ f_e^P : # players taking edge e.

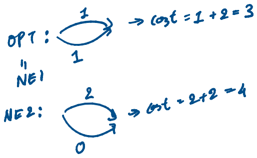
$$cost_i(P) = \sum_{e \in P_i} c_e(\underline{f_e^P}) \qquad cost(P) = \sum_i cost_i(P)$$

NE: P is a NE iff

$$\forall i : \; Cost_i(P) \leq cost_i(Q_i, P_{-i}) \quad \forall Q_i \in \mathcal{P}_i$$

Example:





$$\boxed{PoA = \frac{\text{worst NE cost}}{OPT}} = \frac{4}{3}$$

★ Existence of NE Through Potential Function.

$$\phi : \mathop{\times}_{i=1}^{n} \mathcal{P}_i \to \mathbb{R}_+$$

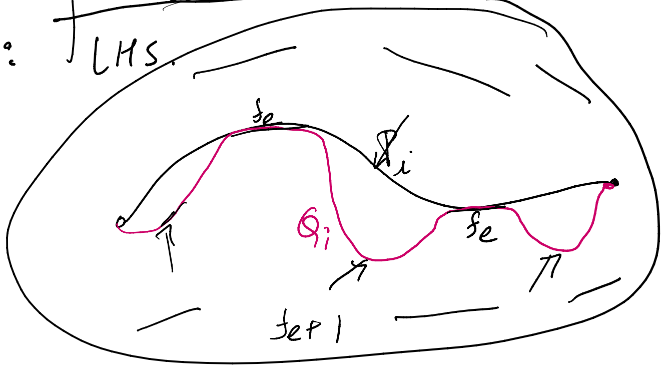
$$\phi(P) = \sum_{e \in E} \sum_{k=1}^{f_e^P} c_e(k)$$

Thm: $\forall i, \forall P, \forall Q_i \in \mathcal{P}_i$ $\qquad f_e = f_e^P$

$$\boxed{\begin{aligned} &Cost_i(P_i, P_{-i}) - Cost_i(Q_i, P_{-i}) \\ &= \phi(P_i, P_{-i}) - \phi(Q_i, P_{-i}) \end{aligned}}$$

Potential Games

Pf: LHS.



$$Cost_i(P_i, P_{-i}) = \sum_{e \in P_i} c_e(f_e)$$

$$- \; \underline{cost_i(Q_i, P_{-i})} = \sum_{e \in Q_i \setminus P_i} c_e(f_e+1) + \sum_{e \in Q_i \cap P_i} c_e(f_e)$$

$$- \sum c_e(f_e+1)$$

$j \neq 1$

$$LHS = \sum_{e \in P_i \setminus Q_i} c_e(f_e) - \sum_{e \in Q_i \setminus P_i} c_e(f_e + 1)$$

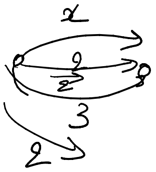
$$RHS = \sum_{e \in E} \sum_{k=1}^{f_e} c_e(k) - \left[\sum_{\substack{e \in E \setminus (P_i \cup Q_i) \\ \cup (P_i \cap Q_i)}} \sum_{k=1}^{f_e} c_e(k) + \sum_{e \in Q_i \setminus P_i} \sum_{k=1}^{f_e+1} c_e(k) + \sum_{e \in P_i \setminus Q_i} \sum_{k=1}^{f_e - 1} c_e(k) \right]$$

$$\boxed{= \sum_{e \in P_i \setminus Q_i} c_e(f_e) - \sum_{e \in Q_i \setminus P_i} c_e(f_e+1)} = LHS$$

★ Consequences: (Potential Game) P^* is a NE. $\boxed{\text{Pure NE}}$

→ $P^* \in \arg\min_{P \in \times_{i=1}^{n} P_i} \phi(P) \Rightarrow P^*$ is a NE.

→ Gives a simple algorithm to reach a NE.



P $\quad \exists i \; cost_i \downarrow \Rightarrow \phi \downarrow$

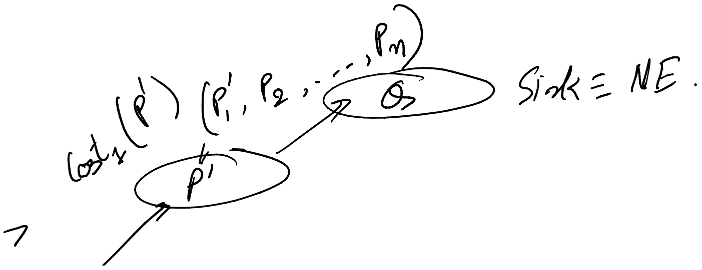
$\phi(P_0) > \phi(P_1) > \phi(P_2) \dots$

$P_0 \xrightarrow{i_1,\ BR} P_1 \xrightarrow{i_2} P_2 \longrightarrow \dots$

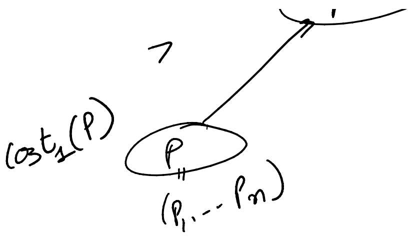
$\boxed{\text{Best Response Dynamics}}$

BRD $\xrightarrow{\text{converges}}$ NE.

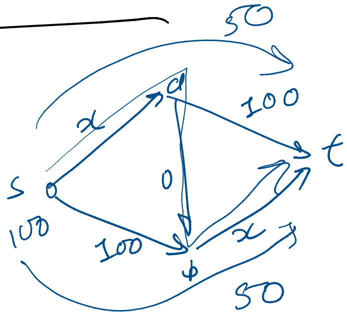
→ Problem of finding Pure NE is in <u>PLS</u>



Sink ≡ NE.

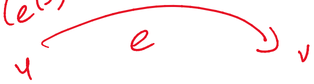


✭ Affine Cost Func. : PoA



Given $G = (V, E)$ directed.

$c_e(x) = a_e x + b_e$ $\qquad a_e, b_e \geq 0$



Thm: PoA w. affine costs $\leq \frac{5}{2}$

Proof: $P = (P_1, \dots, P_n)$ worst NE $\longrightarrow f_e$ = # players taking edge e at NE P

$P^* = (P_1^*, \dots, P_n^*)$ OPT $\longrightarrow f_e^*$ = # players taking edge e at OPT P^*

$$\text{NE} \Rightarrow \forall i: \ \mathrm{cost}_i(P_i, P_{-i}) \overset{?}{\leq} \mathrm{cost}_i(P_i^*, P_{-i})$$

$$\Rightarrow \sum_{e \in P_i} c_e(f_e) \leq \boxed{\sum_{e \in P_i^* \setminus P_i} c_e(f_e + 1) + \sum_{e \in P_i^* \cap P_i} c_e(f_e)}$$

$$\leq \sum_{e \in P_i^*} c_e(f_e + 1) \quad (\because c_e\text{'s are non-dec})$$

$$\mathrm{cost}(P) = \sum_i \mathrm{cost}_i(P) \leq \sum_i \mathrm{cost}_i(P_i^*, P_{-i}) \leq \boxed{\sum_{i=1}^{n} \sum_{e \in P_i^*} c_e(f_e + 1)}$$

$$= \sum_{e \in E} \sum_{i : e \in P_i^*} c_e(f_e + 1) = \sum_{e \in E} c_e(f_e + 1) \cdot f_e^*$$

$$= \sum_{e \in E} f_e^* \left(a_e (f_e + 1) + b_e \right)$$

$$= \sum_{e \in E} a_e \underset{x}{f_e^*} (\underset{y}{f_e + 1}) + \sum_{e \in E} b_e f_e^*$$

$x, y \in \{0, 1, \dots\}$

$e \in E$ x

$$= \sum_{e\in E} a_e\left(\frac{5}{3} f_e^{*2} + \frac{1}{3} f_e^{2}\right) + \sum_{e\in E} b_e f_e^{*}$$

$x, y \in \{0, 1, \dots\}$

$x(y+1) \le \frac{5}{3}x^2 + \frac{1}{3}y^2$

$$\le \sum_{e\in E} \frac{5}{3} a_e f_e^{*2} + \frac{5}{3} b_e f_e^{*} + \frac{1}{3}\sum_{e\in E}\left(a_e f_e^2 + b_e f_e\right)$$

$$= \frac{5}{3}\sum_{e\in E} f_e^*\,(\underbrace{a_e f_e^* + b_e}_{c_e(f_e^*)}) + \frac{1}{3}\sum_{e\in E} f_e\,(\underbrace{a_e f_e + b_e}_{c_e(f_e)})$$

$$= \frac{5}{3}\sum_{e\in E} f_e^*\, c_e(f_e^*) + \frac{1}{3}\sum_{e\in E} f_e \cdot c_e(f_e)$$

$$\text{cost}(P^*) = \sum_i \text{cost}_i(P^*) = \sum_i \sum_{e\in P_i^*} c_e(f_e^*) = \sum_{e\in P_i^*} f_e^*\, c_e(f_e^*)$$

$$= \frac{5}{3}\sum_{i=1}^{n} \text{cost}_i(P^*) + \frac{1}{3}\sum_{i=1}^{n} \text{cost}_i(P)$$

$$\therefore\ \text{cost}(P) \le \boxed{\frac{5}{3}\,\text{cost}(P^*) + \frac{1}{3}\,\text{cost}(P)}$$

$$\Rightarrow \frac{2}{3}\,\text{cost}(P) \le \frac{5}{3}\,\text{cost}(P^*)$$

$$\Rightarrow \text{PoA} = \frac{\text{cost}(P^*)}{\text{cost}(P)} \le \frac{5}{2}$$

☆ High-level steps followed by the above proof.

① P: worst NE, P^*: OPT

② NE $\Rightarrow \underline{\text{cost}(P)} = \sum_i \text{cost}_i(P) \le \boxed{\sum_i \text{cost}_i(P_i^*, P_{-i})}$

③ $\sum \text{cost}_i(P_i^*, P_{-i}) \le \overset{\lambda}{5}\ \text{cost}(P^*) + \overset{\mu}{\frac{1}{3}}\ \text{cost}(P)$

③ $\sum_i \text{cost}_i(P_i^*, P_{-i}) \leq \frac{5}{3} \text{cost}(P^*) + \left(\frac{1}{3}\right) \text{cost}(P)$

②.③ $\Rightarrow \left(1-\frac{1}{3}\right) \text{cost}(P) \leq \frac{5}{3} \text{cost}(P^*) \Rightarrow \text{PoA} = \frac{\text{cost}(P)}{\text{cost}(P^*)} \leq \frac{\overset{\rightarrow \lambda}{5/3}}{\underset{\downarrow (1-\mu)}{\left(1-\frac{1}{3}\right)}}$

$= \frac{\lambda}{1-\mu}$

★ Any Game : $\{1 \dots n\}$ players

player i's strategy set S_i . $s_i \in S_i$

$s = (s_1, \dots, s_n)$ NE

$s^* = (s_1^*, \dots, s_n^*)$ OPT

NE $\Downarrow$

$$\text{cost}(s) \leq \boxed{\sum_{i=1}^{n} \text{Cost}_i(s_i^*, s_{-i}) \leq \lambda \,\text{cost}(s^*) + \mu \,\text{cost}(s)}$$

(λ, μ) - smooth Game.

$\Rightarrow \text{PoA} \leq \frac{\lambda}{1-\mu}$

★ (λ, μ) - smooth Game :

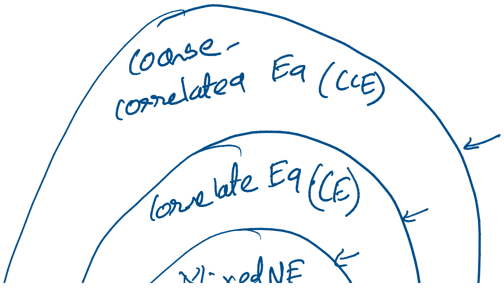
$\forall s, s^* \in \times_{i=1}^{n} S_i$

$$\sum_i \text{Cost}_i(s_i^*, s_{-i}) \leq \lambda \,\text{cost}(s^*) + \mu \,\text{cost}(s)$$

$\Downarrow$

$$\boxed{\text{PoA}^{CCE} \leq \frac{\lambda}{1-\mu}}$$

$\text{PoA}^{PNE} \leq \text{PoA}^{MNE} \leq \text{PoA}^{CE} \leq \text{PoA}^{CCE} \leq \frac{\lambda}{1-\mu}$



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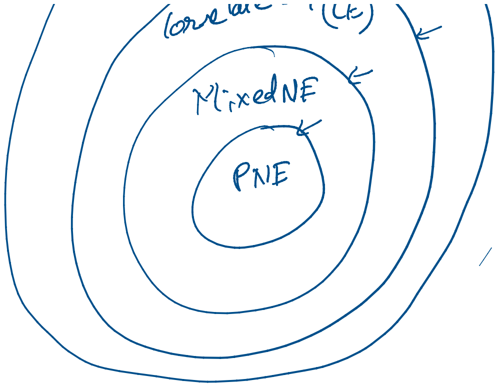
Correla (CE)
MixedNE
PNE