Lecture 13

Thursday, October 8, 2020 1:56 PM

(Recall) Security Games

→ Set of targets $[n] = \{1 \dots n\}$

→ Two players

Defender

Strategy set of defender: $\mathcal{E} \subseteq \{0,1\}^n$

$e \in \mathcal{E}$, $e_i = 1 \equiv$ target i is protected as per e.

Play: (e, i)

$r_i e_i + (1 - e_i) c_i$

- $r_i e_i$: reward if i is protected
- $(1-e_i) c_i$: cost if i is not protected

$r_i \geq 0$, $c_i \leq 0$

Random strategy: (p, y), $p \in \Delta(\mathcal{E})$, $y \in \Delta([n])$

Attacker

Strategy set: $[n]$

$i \in [n]$

reward if i not protected

$s_i (1 - e_i) + \xi_i e_i$

cost if i is protected

$s_i \geq 0$, $\xi_i \leq 0$

$y \in \Delta([n])$

y_i: Prob that i is attacked.

Expected Payoffs:

$$\sum_{e \in \mathcal{E}} p_e \sum_{i \in [n]} y_i \left(r_i e_i + (1 - e_i) c_i \right)$$

$$= \sum_{i \in [n]} y_i \left[r_i \left(\sum_{e \in \mathcal{E}} p_e e_i \right) + c_i \left(1 - \sum_{e \in \mathcal{E}} p_e e_i \right) \right]$$

$x_i = \sum_{e \in \mathcal{E}} p_e \cdot e_i \leq \sum_{e \in \mathcal{E}} p_e = 1$

$x_i \leq 1$?

↓ marginal probability that target i is defended. (Prob. with which target i is protected)

$$= \sum_{i \in [n]} y_i \left(r_i x_i + c_i (1 - x_i) \right)$$

$$= \sum_{i \in [n]} y_i \left(s_i (1 - x_i) + \xi_i x_i \right)$$

$$= \sum_{i \in [n]} y_i \left(x_i (r_i - c_i) \right) + \sum_{i \in [n]} c_i y_i$$

$x \in P = \left\{ \sum_{e \in \mathcal{E}} p_e \cdot e \;\middle|\; p \in \Delta(\mathcal{E}) \right\}$

$$= \sum_{i \in [n]} x_i \left(y_i (r_i - c_i) \right) + \sum_{i \in [n]} c_i y_i$$

Defenders' Best Response (DBR): Fix $y \in \Delta([n])$ for the Attacker

$$\arg\max_{x \in \mathcal{P}} \sum_{i \in [n]} x_i \left(y_i (r_i - c_i) \right) + \sum_{i \in [n]} c_i y_i \quad \text{(constant given } y\text{)}$$

$$= \arg\max_{x \in \mathcal{P}} \sum_{i \in [n]} x_i \left(y_i (r_i - c_i) \right) \qquad w_i \geq 0$$

$$= \arg\max_{x \in \mathcal{P}} \sum_{i \in [n]} x_i w_i \rightarrow \sum_{e \in \mathcal{E}} p_e \sum_{i \in [n]} e_i w_i$$

$$= \arg\max \sum_{e \in \mathcal{E}} \sum_{i \in [n]} e_i w_i$$

$$= \arg\max_{e \in \mathcal{E}} \langle e, w \rangle$$

($\mathcal{E}$: combinatorial set)

DBR problem:
Given $w \in \mathbb{R}^n$, $w \geq 0$ fixed
$\arg\max_{e \in \mathcal{E}} \langle e, w \rangle$

($|\mathcal{E}|$ could be $\exp(n)$)

Lemma: If there is a polynomial time algo to solve the DBR problem then we can compute stackelberg eq. in polynomial time.

pf: Defender "thinks", what all x will force attacker to attack say target $k \in [n]$

LP_k

$$\max: r_k x_k + c_k (1 - x_k)$$

s.t.

$$\rightarrow s_k (1 - x_k) + \xi_k x_k \geq s_i (1 - x_i) + \xi_i x_i \quad \forall i \neq k$$

$$\forall i \in [n]$$

makes k a B.R. for

$x_i = \sum_{e} p_e \cdot e_i$

Dual va.
y_i (1)
w_i (2)

takes k a B.R. for attacker.

$\Rightarrow s_k(1-\alpha_k) \cdot \gamma_k \le$ ... $x_i = \sum_{e \in \mathcal{E}} p_e \cdot e_i$ ... $\forall i \in [n]$ → ②

→ ③

$\sum_{e \in \mathcal{E}} p_e = 1$

→ ④

$p \ge 0$

$O(n)$ constraints But $O(|\mathcal{E}|)$ many variables $\approx O(2^n)$

Dual LP_k: $\min: \sum_{i \ne k} (s_k - s_i) y_i - v$

s.t. (can be efficiently)

$w_k = (r_k - c_k) - \left(\sum_{i \ne k} y_i\right)(s_k - \xi_{pk})$

$w_i = s_i - \xi_i \quad \forall i \ne k$

$v \ge \langle e, w \rangle \quad \forall e \in \mathcal{E}$ → ⑤

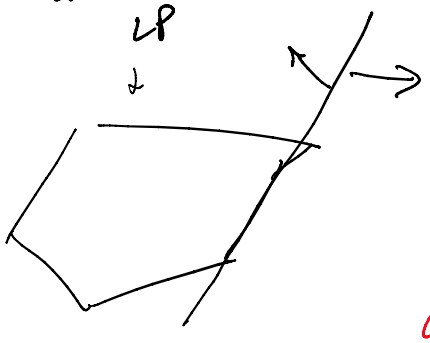
exp. many

$y \ge 0$

Feasible set

★ Seperation Oracle:

Given: Feasible set of an Dual LP, point $(\vec{y}^*, \vec{w}^*, v^*) \to z$



DBR: Given $w \ge 0$

→ find $e^* = \arg\max_{e \in \mathcal{E}} \langle e, w^* \rangle$

$\downarrow e^*$

check if $v^* \ge \langle e^*, w^* \rangle$

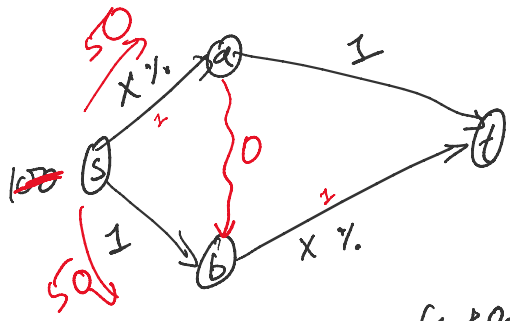
if yes then z is feasible

o.w. it is not feasible.

~~(non-atomic)~~

Routing Games:

→ Braess' Paradox (1968)



OPT = 50-50 $\xrightarrow{\text{cost}}$ $(1+0.5)100 = 1.5 \times 100$

‖

NE

all 100 taking

New NE = s−a−b−t

cost(NE) = 2 · 100

Why is it a NE? switch to s−b−t

cost? 2

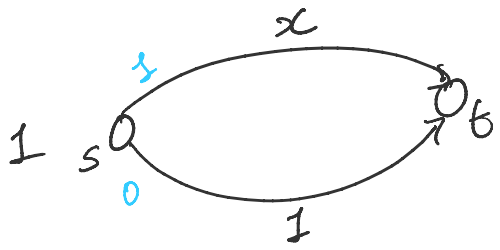
Is 50-50 a NE? NO!

1.5 vs 1.1

New OPT = 50-50

$$\text{Price-of-Anarchy} = \frac{\text{worst NE cost}}{\text{OPT cost}} = \frac{2 \times 100}{1.5 \times 100} = \boxed{\frac{4}{3}}$$

Pigou's Example (1920):



cost: 0.1 x 0.1 + 0.9 x 1

= 0.91

What is NE? 1−0

cost (NE) = 1

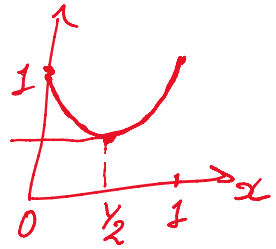
OPT split is x / $(1-x)$

‖

OPT: 50-50

cost (OPT) = $x^2 + (1-x)$

$$\Leftarrow \begin{cases} \frac{d}{dx}\, x^2 - x + 1 = 0 \\ \Rightarrow 2x - 1 = 0 \\ \Rightarrow x = \frac{1}{2} \end{cases}$$

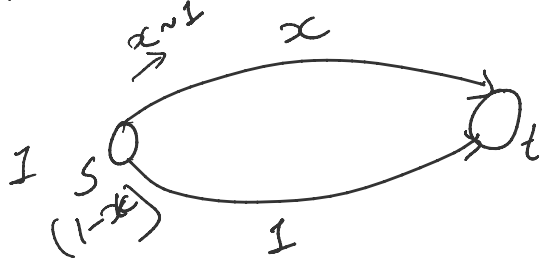


cost (OPT) = $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$$\text{PoA} = \frac{\text{NE cost}}{\text{OPT cost}} = \frac{1}{3/4} = \boxed{\frac{4}{3}}$$

PoA = NE cost / OPT cost → 3/4 [5]

as $p \to \infty$ OPT: $1 \to 1$, $0 \to p$ $\quad p \in \mathbb{Z}^+$



New NE? $\xrightarrow{1}$

cost(NE) = $1 \cdot 1^p = 1$

OPT: x vs $(1-x)$ split $\quad$ cost(OPT) = $x \cdot x^p + (1-x) \cdot 1$

$= x^{p+1} - x + 1$

$\frac{d}{dx}(x^{p+1} - x + 1) = 0$

$\Rightarrow (p+1)x^p - 1 = 0 \Rightarrow x = \left(\frac{1}{p+1}\right)^{\frac{1}{p}}$

cost(OPT) $\xrightarrow{\lim_{p\to\infty}} \left(\frac{1}{p+1}\right)^{\frac{p+1}{p}} - \left(\frac{1}{p+1}\right)^{\frac{1}{p}} + 1 \to 0$

($\to 0$ and $\to 1$)

$$PoA = \frac{\text{NE cost}}{\text{OPT cost}} = \frac{1}{\to 0} \to \infty \quad \text{as } p \to \infty$$

Conclusion: Definitely degree of the cost function matters. Does complexity of n/w also matter? NO!!

Goal.

★ Directed n/w $G = (V, E)$

- s, t nodes
- r units of flow from s to t.
- $c_e \in \mathcal{C}$.

- a fixed δ flow from ...

→ cost function e, $c_e \in \mathcal{C}$.

non-neg, non-dec, continuous.

$$\text{PoA} = \frac{\text{worst NE cost}}{\text{OPT cost.}}$$

Thm: Given a class $\mathcal{C}$ of cost functions, among all n/w with edge costs $c_e \in \mathcal{C}$, "Pigou-like" n/w has the worst PoA.