CS 598RM: Algorithmic Game Theory, Fall 2019
Practice Exam Solutions

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1. Answer the following. Each question is worth 5 points.

   (a) A two player game represented by matrices (A, B) is called constant sum if $A(i,j) + B(i,j) = c$, $\forall i, j$ where $c \in R$. Is the following statement *True* or *False*: The set of Nash equilibria of this game is a convex set.

   (b) In a stable roommate problem there is a set of m dorm rooms each of which can accommodate exactly two students, and a set n of students where $m > n/2$. Each student has a total ordering on all the other students (non-bipartite), and prefers to have a roommate then to live alone. Given an assignment of students to rooms, a pair of students (s_1, s_2) is unstable if they both prefer each other more than their current assignment. An assignment is called stable if there is no unstable pair. Construct an example that has no stable assignment.

   Hint: *Think of a three node graph.*

   (c) Consider the "Knapsack Auction," where there is a knapsack with total weight W, and n bidders each with a public weight w_i and private valuation v_i, for $i = 1, 2, \ldots, n$. The mechanism wishes to select a subset of agents with total weight at most W. A selected agent gets utility v_i, and an unselected agent, 0.

   Consider the greedy allocation mechanism which collects bids from the agents (reported valuations), sorts them in decreasing order of bid-per-weight $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \cdots \geq \frac{b_n}{w_n}$, then allocates the players one by one until there is no room left.

   If there are *two* knapsacks, with weights W_1 and W_2, and the mechanism first allocates greedily up to W_1 in the first bag, then, with the remaining jobs, greedily up to W_2 in the second.

   Is this a monotone allocation rule? If yes, prove it. If no, give a counterexample.

   (d) Give an example of a two-player game with pure Nash equilibria, where best response dynamics can cycle.

Solution.

(a) The statement is true. The reason is as follows. For any constant-sum games, the objective of player 2, of maximizing $x^T By$ given the strategy x of player 1, is equivalent to that of minimizing $x^T Ay$, just as in the special case of zero-sum games. As a consequence, the set of Nash equilibria can be captured by a linear program and its dual, and by the property of LP solutions, the set of solutions is convex.

(b) Consider 3 students A, B, C. A prefers B over C; B prefers C over A, C prefers A over B. If 2 students (or more) are kept alone, the assignment is obviously unstable as both of them prefer each other over being alone. In any assignment which pairs two of them, and keeps the third alone, one of the students in the matched pair prefers the third student, and the third prefers him over being alone, hence, this is unstable as well.

(c) This is a monotone allocation rule. Consider the following cases:

(i) The agent was not allocated at the lower bids.

(ii) The agent was allocated to the 1st knapsack at the lower bids.

(iii) The agent was allocated to the 2nd knapsack at the lower bids.

In case (i), there is nothing to consider, since the agent can not have a worse outcome. In case (ii), the agent can only be considered sooner at higher bids, and so they will be allocated to the 1st knapsack again. In case (iii), raising the agent's bid will have them considered sooner. Either they will be allocated in the 1st knapsack instead of the 2nd, or this raised bid will not have changed the set of agents allocated to the 1st knapsack remains unchanged, and we may view this as identical to case (ii).

(d) Consider the following 2-player game:

10, 10	0, 0	0, 0
0, 0	1, 0	0, 1
0, 0	0, 1	1, 0

Playing the top row and the left column is a PNE since no player benefits from deviating. However, if we are in the bottom right submatrix, the row and column players will take turns alternating between the 2nd and 3rd row and column, thus cycling.

2. Consider a game where n players are allocating a shared bandwidth of 1. Each player i chooses an amount $1 \geq x_i \geq 0$ and the utility of player i is $U_i(x_i, \boldsymbol{x}_{-i}) = x_i \cdot (1 - \sum_{j=1}^{n} x_j)$. Note that if the sum of x_j's is greater than 1, then all players have a negative utility.

   (a) Show that this game has a Nash equilibrium, and give a Nash equilibrium for this game.

   (b) Is the Nash equilibrium unique? Prove your answer.

   (c) Is it a potential game? Prove your answer.

Solution.

$(a), (b)$ At an equilibrium, if $\sum_{j=1}^{n} x_j > 1$, there must be a player i, s.t. $x_i > 0$ — i gets a negative utility, whereas switching to $x_i = 0$ would give it a better utility of 0, a contradiction to the NE definition. Hence, $\sum_{j=1}^{n} x_j \leq 1$ at any NE. Further, fixing x_{-i}, U_i as a function of x_i is a strictly concave function, and has a unique maxima at $x_i = (1 - \sum_{j \neq i} x_j)/2$ which is required for x_i to be a best response at any NE. Since this holds for any i, we have,

$$x_i = (1 - \sum_{j \neq i} x_j)/2 \quad \forall i$$

$$2x_i = (1 - \sum_{j \neq i} x_j) \quad \forall i$$

$$x_i = (1 - \sum_{j} x_j) \quad \forall i$$

Hence, all x_i's are equal. using the last equation above, it implies, $x_i = 1/(n+1)$ for each i. Thus, this is a unique equilibrium. (Also showing the existence from the first part. To just show the first part, one can use the fact that the game is symmetric and must have a symmetric equilibrium, which directly leads to the last step of this proof.)

(a) Let s denote $\sum_j x_j$. When i plays x_i, its payoff is

$$U_i(x_i, x_{-i}) = x_i(1 - s)$$

If i switches to $x_i + c$, its payoff changes by

$$(x_i + c)(1 - s - c) - x_i(1 - s) = c(1 - s - x_i) - c^2$$

Subsequently, if some j now switched from x_j to $x_j + d$, its payoff change can be computed to be

$$(x_j + d)(1 - s - c - d) - x_j(1 - s - c) = d(1 - s - c - x_j) - d^2$$

Adding the two, we get,

$$(1 - s)(c + d) - cx_i - dx_j - (c^2 + cd + d^2)$$

Thus, this expression remains unchanged, if j switched from x_j to $x_j + d$ first, and then i switched from x_i to $x_i + c$. This sum of change in respective payoffs in a sequence of unilateral deviations, is independent of the order deviations if and only if the game is a potential game. Hence, this game is a potential game. The exact potential can be found to be $\Phi(x_1, \ldots, x_n) = \sum_k x_k(1 - x_k) + \frac{1}{2}\sum_{j \neq k} x_j x_k$ and, verifying that it's indeed a potential follows from definition of a potential.

3. Consider a variant of the knapsack auction in which both the valuation v_i and the size w_i of each bidder i are private. A mechanism now receives both bids $\boldsymbol{b} = (b_1, \ldots, b_n)$ and reported sizes $\boldsymbol{a} = (a_1, \ldots, a_n)$ from the bidders, where n is the number of bidders. Assume each v_i, w_i, b_i, a_i to be positive. An allocation rule $x(\boldsymbol{b}, \boldsymbol{a})$ now specifies the amount of capacity allocated to each bidder, as a function of the bids as well as the reported sizes. Feasibility of an allocation requires that the total capacity allocated should not exceed the total capacity W of the shared resource, i.e., $\sum_{i=1}^{n} x_i(\boldsymbol{b}, \boldsymbol{a}) \leq W$ for every $\boldsymbol{b}$ and $\boldsymbol{a}$. We define the utility of a bidder i as $v_i - p_i(\boldsymbol{b}, \boldsymbol{a})$ if she gets her desired capacity (i.e., $x_i(\boldsymbol{b}, \boldsymbol{a}) \geq w_i$) and as $-p_i(\boldsymbol{b}, \boldsymbol{a})$ otherwise. Note that this is not a single-parameter environment.

   Consider the following mechanism. Given bids $\boldsymbol{b}$ and reported sizes $\boldsymbol{a}$, the mechanism runs the greedy knapsack auction: It first sorts the bidders in the decreasing order of $\frac{b_i}{a_i}$. Then moving from left to right in the sorted list, when it is bidder i's turn it awards her a capacity equal to her reported size a_i if a_i amount of space is available in the sack ($x_i(\boldsymbol{b}, \boldsymbol{a}) = a_i$) and subtracts a_i from the sack space, otherwise i gets no allocation ($x_i(\boldsymbol{b}, \boldsymbol{a}) = 0$). And it continues allocating this way until the end of the list is reached. Every *winning bidder* i (i.e., one which does get allocated its a_i) is charged a price of p_i defined as "the minimum bid which allows i to be a winning bidder, by fixing all the bids b_{-i}, reported sizes a_{-i} and its own a_i". A losing bidder neither gets anything, nor pays anything.

   A truthful strategy for a player i in this case is one, where $b_i = v_i$, as well as $a_i = w_i$. Is this mechanism DSIC? If yes, prove so, otherwise give an explicit counterexample where it is not DSIC.

Solution.

Consider an arbitrary bidder i. Fix any b_{-i}, and a_{-i}. We now show, that reporting the true v_i and w_i gives i the maximum utility. When the bids are $(b_i = v_i, b_{-i})$, and reported sizes are $(a_i = w_i, a_{-i})$, the payment p_i^* for i is as follows: $p_i^* = 0$ if i doesn't get allocated, leading to a utility of 0. If it does, let bidder j be the one with the highest b_j/a_j $(= r_j$, say) smaller than $b_i/a_i (= v_i/w_i)$ such that j does not get allocated anything in presence of i, but does get allocation when i is not there. Thus, i gets allocation as long as its $r_i = b_i/a_i$ beats r_j (keeping $a_i = w_i$ fixed). Thus, the payment (given by the critical bid), is $p_i^* = r_j \cdot w_i$ — the resultant utility for i is $v_i - p_i^* = v_i - r_j \cdot w_i > 0$, since $r_j < v_i/w_i$ by definition. Thus, reporting both truthfully gives a non-negative utility. Now, suppose i wants to deviate to untruthful b_i, a_i. Consider the following cases:

Case 1: $a_i < w_i$. In this case, the utility is never positive by definition of the utility, and so, there's no incentive to report untruthful $a_i < w_i$ (regardless of b_i).

Case 2: $a_i > w_i$. If i wasn't allocated when reporting truthfully, there will be an incentive switching to this case only if it gets some allocation here. Define $r_i = b_i/a_i$. Then, i needs to have r_i large enough to get some allocation, say at least r_0. Clearly, i did not have it when bidding truthfully, hence, $r_0 > v_i/w_i$. If r_0 is beaten by i by reporting b_i, a_i, that means, $r_0 \leq b_i/a_i$. Hence, the payment is $r_0 * a_i > r_0 * w_i > v_i$, leading to a negative utility. Hence, i does not want to deviate in this case. On the other hand, if i was allocated, when bidding truthfully, it still needs to beat r_j as computed above, since having $r_i < r_j$ would not have gotten i any allocation by reporting w_i, so it wouldn't get any allocation by having $r_i < r_j$

and reporting $a_i > w_i$. Thus, $r_i = b_i/a_i > r_j$. Hence, payment $p_i \geq a_i * r_j > w_i * r_j = p_i^*$. Hence, the payment increases, and the utility decreases. Hence, i does not deviate to any untruthful reporting in this case as well.

Hence, reporting true v_i and w_i is a best response to any arbitrary b_{-i}, and a_{-i}, for any arbitrary i, making the mechanism DSIC by definition.

4. Consider a load balancing game with n jobs and m machines. Each job is a player who chooses a machine to run on, and trying to minimize its *completion time*. Job j has size p_j, and any jobs can choose any of the m machines. Let $r_i(x)$ be the time needed by machine i to process the total load (sum of sizes of assigned jobs) is x. Assume $r_i(x) = x$ for all machines. A machine releases a job only after finishing all of its jobs, i.e., if the set of jobs that choose machine j is $S \subseteq \{1, \ldots, n\}$, then completion time of job $j \in S$ is $\sum_{j \in S} p_j$.

   (a) Is this a potential game? Prove your answer.

   (b) Suppose the social welfare is given by the maximum completion time. Show that the Price of Anarchy is upper-bounded by 2.

Solution.

(a) No, this is not a potential game. To prove contradiction, suppose there exists a potential function ϕ, in a simple game involving 2 jobs of unequal sizes and 2 machines. Thus, both players have only 2 pure strategies. Let s_{ij} denote the joint strategy in which job 1 is on machine i, and job 2 is on machine j. Let $t_i(s)$ denote the completion time of job i when the joint strategy is s. Consider the joint strategy s_{ij}. If job 1 switches to machine i', the change in its respective costs(completion times) is $t_1(s_{i'j}) - t_1(s_{ij})$ which must equal $\phi(s_{i'j}) - \phi(s_{ij})$. Thus, if both jobs are on machine 1, and job 1 switches to machine 2, we get,

$$\begin{aligned} \phi(s_{21}) - \phi(s_{11}) &= t_1(s_{21}) - t_1(s_{11}) \\ &= p_1 - (p_1 + p_2) \end{aligned} \tag{1}$$

similarly, if job 2 also switches to machine 2, we can write,

$$\begin{aligned} \phi(s_{22}) - \phi(s_{21}) &= t_2(s_{22}) - t_2(s_{21}) \\ &= (p_1 + p_2) - p_2 \end{aligned} \tag{2}$$

Adding (1) and (2) gives,

$$\phi(s_{22}) - \phi(s_{11}) = p_1 - p_2 \tag{3}$$

Now, if starting from the joint strategy s_{11}, if we reverse the order and make job 2 switch to machine 2 first, followed by job 1 switching to machine 2, we can write equations as above to get,

$$\phi(s_{22}) - \phi(s_{11}) = p_2 - p_1 \tag{4}$$

Thus, (3) and (4) imply that the jobs have equal sizes, a contradiction. Thus, we have used the fact that in a potential game, if the change in costs/payoffs via a unilateral deviation is summed over a sequence of deviations, this sum must be independent of the order(permutation) of the deviations.

(b) Consider any equilibrium. Let the social welfare at this equilibrium be x. That is, a machine i (say), has a load of x, which is maximum among all the machines. We will prove the maximum completion time is at least $x/2$ in any other joint-strategy, which will give us the desired upper bound on PoA. Consider two cases:

Case I: i has only one job

Thus, the only job on i, say job j, has a size x. In any joint-strategy, and in particular the one with optimum social welfare, the machine which has j has a total load of at least x. So, its completion time, and consequently, the maximum completion time too, must be at least x.

Case II: i has at least two jobs

As the total load on i is x, there must be a job on i, say job j, of size y that is at most $x/2$. Since we have an equilibrium, job j does not want a different machine, which means that every other machine must have a load of at least $x - y$, hence, at least $x/2$. Thus, the total sum of sizes is at least $x + (n-1)x/2$. In any joint strategy, when this total load is distributed across machines, there must be at least one machine with a load of more than $x/2$, and thus, the maximum completion time is at least $x/2$.

5. Consider the following weighted generalization of the network cost-sharing game. For each player i, we have a weight $w_i > 0$. As before, each player selects a single path connecting her source and sink. But instead of sharing edge cost equally, players are now assigned cost shares in proportion to their weight. In particular, for a strategy vector S and edge e, let S_e denote those players whose path contains e, and let $W_e = \sum_{i \in S_e} w_i$ be the total weight of these players. Then player i pays $c_e \cdot \frac{w_i}{W_e}$ for each edge $e \in P_i$. Note that if all players have the same weight, this is the original game.

   (a) Show that, in general, this game does not have an exact potential function.

   (b) Show that there exists a potential function Φ such that,

$$\forall i, \forall \text{ paths } P_i, P_i' : \quad \Big(\Phi(P_i, P_{-i}) - \Phi(P_i', P_{-i})\Big) \cdot \Big(C_i(P_i, P_{-i}) - C_i(P_i', P_{-i})\Big) \geq 0$$

$$\forall i, \forall \text{ paths } P_i, P_i' : \quad \Phi(P_i, P_{-i}) = \Phi(P_i', P_{-i}) \quad \iff \quad C_i(P_i, P_{-i}) = C_i(P_i', P_{-i})$$

   (c) Using the above, show that the game has a pure Nash equilibrium.

Solution.

(a) Consider a 2-node-2-edge-2-player network where both edges are $s \to t$ and have total cost 1, and both players are routing from s to t. Suppose $w_1 = 1$ and $w_2 = 2$. Suppose for a contradiction that there exists an exact potential function Φ. Then,

$$\begin{aligned}\Phi(1,1) - \Phi(2,2) &= \big(\Phi(1,1) - \Phi(1,2)\big) + \big(\Phi(1,2) - \Phi(2,2)\big) \\ &= (\tfrac{2}{3} - 1) + (1 - \tfrac{1}{3}) \ = \tfrac{1}{3}\end{aligned}$$

However,

$$\begin{aligned}\Phi(1,1) - \Phi(2,2) &= \big(\Phi(1,1) - \Phi(2,1)\big) + \big(\Phi(2,1) - \Phi(2,2)\big) \\ &= (\tfrac{1}{3} - 1) + (1 - \tfrac{2}{3}) \ = -\tfrac{1}{3}\end{aligned}$$

a contradiction.

(b) Consider the potential function $\Phi(P) = \sum_{e:e \in P}(c_e / W_e)$, where W_e is the total weight of players on e. If player i is deviating from path P_i to path P_i', then the difference in their cost is

$$\Delta C_i := \sum_{e \in P_i' \setminus P_i} \frac{w_i \cdot c_e}{W_e + w_i} - \sum_{e \in P_i \setminus P_i'} \frac{w_i \cdot c_e}{W_e}$$

where W_e is the total weight before the deviation. Observe that the difference in Φ is exactly the same amount, without the w_i term. Therefore, $\Phi(P_i', P_{-i}) - \Phi(P_i, P_{-i}) = \frac{1}{w_i}\big(C_i(P_i', P_{-i}) - C_i(P_i, P_{-i})\big)$. Since w_i is a positive constant, the two desired conditions must hold.

Incorrect Solution. One could suggest the following as the potential function:

$$\sum_{e \in P} \sum_{i: e \in P_i} \frac{w_i \cdot c_e}{W_e}.$$

Observe, however, that this function is equal to $\sum_{e \in P} c_e$, and therefore cannot be correct.

(c) To find a PNE, we simply begin with any assignment of path to the n players, and whenever there is a possible improvement, we take this improvement. Since the improvement is a net benefit to the deviating player, this will strictly decrease the potential function from part (b). Since there are only finitely many possible path assignments, there can only be finitely many values that the potential function will ever take, and this procedure must eventually terminate at an assignment of paths where no player can benefit from deviating: a PNE.