CS 598RM: Algorithmic Game Theory, Fall 2019
HW 2 (due on Sunday, 20th Oct at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.

2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.

3. Please type your solutions if possible in Latex or doc whatever is suitable. We will upload submission instructions on the course webpage and Piazza.

4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.

5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.

6. No late assignments will be accepted.

7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

1. (a) (5 points) Consider a first price single item auction with set of bidders A. Let $n = |A|$ and the private value v_i of each bidder i come from a uniform distribution $[0, 1]$. Show that the bid profile $(b_1, \ldots, b_n)$, where $b_i = \frac{(n-1)}{n} v_i$, $\forall i \in A$ is a Nash equilibrium.

   Hint: *Show that when $b_j = \frac{(n-1)}{n} v_j, \forall j \neq i$, then $b_i = \frac{(n-1)}{n} v_i$ is the best response of bidder i.*

   (b) (5 points) Consider a single item second-price auction with n bidders. Assume that a subset S of the bidders decide to collude, that is, the bidders in S coordinate to submit false bids in order to maximize the sum of their payoffs. Under what conditions will this collusion be successful, and what should their collective strategy be?

   Solution.

2. Recall the sponsored search auction problem discussed in class: there are k slots, the jth slot has a known click-through rate (CTR) of α_j (non-increasing in j), and the payoff of bidder i in slot j is $\alpha_j(v_i - p_j)$, where v_i is the (private) value-per-click of the bidder and p_j is the price charged per-click in that slot. For historical reasons, modern search engines do not use the truthful auction discussed in class. Instead, they use auctions derived from the Generalized Second-Price (GSP) auction, defined as follows:

(1) Rank advertisers by bid; assume without loss of generality that $b_1 \geq b_2 \geq \cdots \geq b_n$.

(2) For $i = 1, 2, \ldots, k$, assign the ith bidder to the ith slot.

(3) For $i = 1, 2, \ldots, k$, charge the ith bidder a price of b_{i+1} per click.

(a) (1 points) Prove that for $k = 2$ and any sequence $\alpha_1 \geq \alpha_2 > 0$, there exist valuations for the bidders such that the GSP auction is not truthful. (This in fact holds for any $k \geq 2$).

(b) (2 points) Fix CTRs for slots and valuations-per-click for bidders. We can assume that $k = n$ by adding dummy slots with zero CTR (if $k < n$) or dummy bidders with zero valuation (if $k > n$). A bid vector b is an equilibrium of GSP if no bidder can increase its payoff by changing its bid. Verify that this translates to the following conditions, assuming that $b_1 \geq b_2 \geq \cdots \geq b_n$: for every i and higher slot $j < i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_j);$$

and for every lower slot $j > i$,

$$\alpha_i(v_i - b_{i+1}) \geq \alpha_j(v_i - b_{j+1}).$$

Hint: *Derive these by adopting i's perspective and "targeting" the slot j.*

(c) (3 points) A bid vector $\mathbf{b}$ with $b_1 \geq \cdots \geq b_n$ is envy-free if for every bidder i and slot $j \neq i$,

$$\alpha_i(v_i - p_i) \geq \alpha_j(v_i - p_j);$$

where $p_j = b_{j+1}$. Verify that an envy-free bid vector is necessarily an equilibrium. (The terminology "envy-free" stems from the following interpretation: for the current price-per-click is p_j of slot j; then the above inequalities say: "each bidder i is as happy getting its current slot at its current price as it would be getting any other slot and that slot's current price".)

(d) (4 points) Prove that, for every set of α_js and v_is, there is an equilibrium of the GSP auction for which the outcome (i.e., the assignment of bidders to slots) and the prices paid precisely match those of the truthful auction discussed in class. If you want, you can assume that the CTRs are strictly decreasing.

Hint: *Recall that using Myerson's lemma we know a closed-form solution for the payments made by the truthful auction (see (8) and (9) in `http://theory.stanford.edu/~tim/f13/l/l3.pdf`). What bids would yield these payments in a GSP auction? Part (c) might be useful for proving that they form an equilibrium.*

Solution.

3. Consider combinatorial auctions for m items among n bidders. The goal of this exercise is to study special cases in which the VCG mechanism can be implemented efficiently. Suppose that each buyer sends as a bid a valuation vector of $2^m - 1$ numbers (a value for each subset of items, since $v_i(\emptyset) = 0$ for all i).

   (a) (2 points) Suppose that $m = 2$. Show that the VCG mechanism can be implemented in time $O(n^2)$. Provide both the allocation and the payments computed by the VCG mechanism.

   (b) (3 points) Suppose that $m = k$, for some fixed $k > 0$. Show that the VCG mechanism can be implemented in time $O(n^2 \cdot f(k))$, for some function f, where $f(k)$ does not depend on n. Notice that for fixed k, this is polynomial in n. Provide both the allocation and the payments computed by the VCG mechanism.

   (c) (5 points) Give an asymptotic upper bound on the number of items m (with respect to the number of agents n) such that the VCG mechanism can be implemented in time that is polynomial in n. In particular, show that the VCG mechanism can be implemented in time $O(n^2 \cdot f(m))$, for some function f, where $f(m)$ does not depend on n, and give an asymptotic upper bound on m such that this running time is polynomial in n.

   Hint: *Use dynamic programming.*

   Solution.

4. In a procurement auction with single-minded bidders, a single buyer (auctioneer) needs to buy a set M of m items from n possible suppliers (bidders). Each supplier i can provide a publicly known single set of items S_i for a privately known price v_i. The buyer needs to buy all items, and aims to minimize the total price paid. Let $S = \{S_1, \ldots, S_n\}$. Consider the following greedy algorithm $\mathcal{A}$ for the problem:

Algorithm 1: $\mathcal{A}(m, n, S)$

---
1 $R \leftarrow M$;
2 $W \leftarrow \emptyset$;
3 **repeat**
4 $j \leftarrow \arg\max_k \frac{|R \cap S_k|}{v_k}$;
5 $R \leftarrow R \setminus S_j$;
6 $W \leftarrow W \cup \{j\}$
7 **until** $R = \emptyset$;
    Result: W

Let v^* denote the total price paid in an optimal solution, R_i denote the set of remaining items after the i-th step of the repeat loop (lines 3 - 7), S_{j_i} denote the set of items that algorithm $\mathcal{A}$ selects at the i-th step, v_{j_i} denote the price of set S_{j_i}, $x_i = \frac{|R_{i-1} \cap S_{j_i}|}{v_{j_i}}$ denote the items-per-unit of cost procured by the buyer at the i-th step and z_i denote the number of remaining items left to procure after the i-th step.

(a) (2 points) Show that for $i \geq 0$

$$x_{i+1} \geq \frac{z_i}{v^*}$$

(b) (2 points) Show that for $i \geq 0$

$$z_i \leq m \cdot \exp\left(\sum_{k=1}^{i} -\frac{v_{j_k}}{v^*}\right)$$

(c) (2 points) Combine items (a) and (b) to show that $\mathcal{A}$ finds a $(1 + \ln m)$-approximation to the optimal procurement.

(d) (4 points) Deduce an incentive-compatible polynomial-time $(1 + \ln m)$-approximation mechanism for procurement auctions among single-minded bidders.

   Hint: *Notice that since this is a procurement auction, the allocation that maximizes the social welfare is the one that "costs the least". Show first that the procurement scheme defined by $\mathcal{A}$ is monotone decreasing (i.e. a winning bid (b_i) is still a winning bid for any $b_i' \leq b_i$), and identify the "critical values" to be paid to the winning suppliers.*

Solution.

5. **(A fun bonus question)** (10 points) Describe a real-life mechanism that you are not satisfied with, and suggest how to make it better.

Solution.

The remaining problems are for self study. Do NOT submit for grading.

- This problem considers variations on the Bulow-Klemperer Theorem (Lecture 21). Consider selling $k \geq 1$ identical items (with at most one given to each bidder) to bidders with valuations drawn iid from F, where F is a regular distribution (i.e., corresponding ϕ^{-1} is monotonically increasing function). Prove that for every $n \geq k$, the expected revenue of the Vickrey auction (with no reserve) with $n + k$ bidders is at least that of the Myerson's optimal auction for F with n bidders.

  (Hint: Myerson's optimal auction will be Vickrey with reserve $\phi^{-1}(0)$, *i.e.,* discard bids below $\phi^{-1}(0)$, give the item to k highest bidders and charge them $\max\{(k+1)^{th}\text{highest bid}, \phi^{-1}(0)\}$)

- Consider a combinatorial auction in a which a bidder can submit multiple bids under different names, unbeknownst to the mechanism. The allocation and payment of a bidder is the union and sum of the allocations and payments, respectively assigned to all of its pseudonyms. Show that the following is possible: a bidder in a combinatorial auction can earn higher utility from the VCG mechanism by submitting multiple bids than by bidding truthfully.

  Can this ever happen in the Vickrey auction? Give a brief explanation.

- Consider the reverse auction we briefly talked about in class: A denotes the set of bidders who are willing to sell the spectrum they hold. There is a set $F \subseteq 2^B$ of feasible sets that is upward closed (i.e., supersets of feasible sets are again feasible), i.e., if $S \in F$ then we can repack S in available range if spectrum of $A \setminus S$ is acquired.

  - Initialize $S = A$.
  - While there is a bidder $i \in S$ such that $S \setminus \{i\}$ is feasible:
      (*) Delete some such bidder i from S such that $S \setminus \{i\}$ is feasible.
  - Return S.

  Suppose we implement the step (*) using a scoring rule, which assigns a number to each bidder i. At each iteration, the bidder with the largest score (whose deletion does not destroy feasibility of S) gets deleted. The score assigned to a bidder can depend on i, is bid, the bids of other bidders that have already been deleted, the feasible set F, and the history of

what happened in previous iterations. (Note a score is not allowed to depend on the value of the bids of other bidders that have not yet been deleted.) Assume that the scoring rule is increasing – holding everything fixed except for b_i, is score is increasing in its bid b_i. Then, show that the allocation rule above is monotone: for every i and b_{-i}, if i wins with bid bid b_i then she will keep winning with any bid less than b_i.