Lec2 (Fri, Jan 20)

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✱ Recap: Two-player Games

Two-players, each with finitely many strategies. Say S_1 & S_2 are the strategy sets of Player-1 (P1) & player-2 (P2) respectively. Let $m = |S_1|$, $n = |S_2|$.

If P1 plays $i \in S_1$, P2 plays $j \in S_2$ then the payoff they get are respectively A_{ij} & B_{ij}. Thus such a game can be represented by two $m \times n$ dimensional matrices

$$\begin{array}{c|c} & 1 \cdots j \cdots n \\ \hline \begin{matrix}1\\ \vdots\\ i\\ \vdots\\ m\end{matrix} & \begin{bmatrix} & & \\ & A_{ij} & \\ & & \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & 1 \cdots j \cdots n \\ \hline \begin{matrix}1\\ \vdots\\ i\\ \vdots\\ m\end{matrix} & \begin{bmatrix} & & \\ & B_{ij} & \\ & & \end{bmatrix} \end{array}$$

Players may want to randomize, so let Δ_1 & Δ_2 be the set of probability distributions over S_1 & S_2 respectively. If P1 plays as per $x \in \Delta_1$ & P2 as per $y \in \Delta_2$ then over S_1 & S_2 ...

as per $x \in \Delta_1$ & P2 as per $y \in \Delta_2$ then

then expected payoffs are

For P1: $\displaystyle\sum_{\substack{i\in S_1\\ j\in S_2}} \Pr[(i,j) \text{ is played}]\, A_{ij} = \sum_{i,j} (x_i \cdot y_j) A_{ij}$

$$= \sum_{ij} x_i A_{ij} y_j = x^T A y$$

Similarly for P2 it is $x^T B y$

☆ Example 1: Battle-of-Sexes.

$S_1 = S_2 = \{F, S\}$

$$A = \begin{array}{c|cc} & F & S \\ \hline F & 1 & 0 \\ S & 0 & 2 \end{array} \qquad B = \begin{array}{c|cc} & F & S \\ \hline F & 2 & 0 \\ S & 0 & 1 \end{array}$$

$x = (1/2, 1/2)$ $y = (1/2, 1/2)$ gives

P1 & P2 both $3/4$.

☆ Nash Equilibrium (NE)

(x, y) is a NE iff

$$x \in \operatorname*{argmax}_{\tilde{x} \in \Delta_1} \tilde{x}^T A y$$

$$x \in \underset{\tilde{x} \in \Delta_1}{\operatorname{argmax}} \; \tilde{x}^T A y$$

$$y \in \underset{\tilde{y} \in \Delta_2}{\operatorname{argmax}} \; x^T A \tilde{y}$$

☆ Example cont^d.

→ $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ is NE?

NO! because P1 can deviate to $(0, 1)$ & get 1 instead of $\frac{3}{4}$.

→ How about $((\frac{1}{2}, \frac{1}{2}), (\frac{2}{3}, \frac{1}{3}))$?

Still no! Because although P1 is fine P2 can deviate to $(1, 0)$ & get 1 instead of $\frac{3}{4}$.

→ How about $((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}))$

YES!

☆ NE Characterization:

Lemma 1: (x, y) is NE iff

① $\forall i \in S_1$, $x_i > 0 \Rightarrow (Ay)_i = \max_k (Ay)_k$

$\Rightarrow (xB) = \max (x^T B)$.

① $\forall i \in S_1$, $x_i > 0 \Rightarrow (Ay)_i = \max_k (Ay)_k$

② $\forall j \in S_2$, $y_j > 0 \Rightarrow (x^T B)_j = \max_k (x^T B)_k$

Proof: $x^T A y = \sum_i x_i (Ay)_i \leq \max_k (Ay)_k$

$\left(\because \sum_i x_i = 1, x \geq 0\right)$

(x,y) being NE $\Rightarrow$ $x^T A y = \max_k (Ay)_k \Rightarrow$ ①

① $\Rightarrow$ $x^T A y = \sum_i x_i (Ay)_i = \max_k (Ay)_k \left(\sum_i x_i\right)$

$= \max_k (Ay)_k \Rightarrow$ NE.

Similar argument follows for ②. ∎

The above lemma gives an easy way to check if the given profile is a NE. Next we will see how to use it to enumerate all NE.

✩ Enumerating all NE.

Lemma 2: Given $T_1 \subseteq S_1$, $T_2 \subseteq S_2$ checking if $\exists$ NE (x,y) s.t. $\{i \mid x_i > 0, i \in S_1\} = T_1$ & $\{j \in S_2 \mid y_j > 0\} = T_2$ can be done by solving a feasibility LP.

by solving a feasibility ...

Proof: The feasibility LP with variables π_1, π_2, x_i's & y_j's is:

$$LP(T_1,T_2)\begin{cases} \forall i \in S_1: \quad x_i \geq 0 \\ \forall j \in S_2: \quad y_j \geq 0 \\ \sum_{i\in S_1} x_i = 1, \quad \sum_{j \in S_2} y_j = 1 \\ \forall i \in S_1: \; \pi_1 \geq (Ay)_i \\ \forall i \in T_1: \; \pi_1 = (Ay)_i \\ \forall j \in S_2: \; \pi_2 \geq (x^T B)_j \\ \forall j \in T_2: \; \pi_2 = (x^T B)_j \end{cases}$$

Ensures $x \in \Delta_1$, $y \in \Delta_2$

π_1 captures $\max_k (Ay)_k$

π_2 captures $\max_k (x^T B)_k$

■

This gives us following algorithm to enumerate all NE.

(1) $\forall T_1 \subseteq S_1$, $\forall T_2 \subseteq S_2$ check if $LP(T_1, T_2)$ is feasible. If yes then output it's solutions.

★ Nash's Existence Proof (1951)

Brouwer's Fixed-point Theorem (BFT):

Brouwer's Fixed-point Theorem (BFT):
Let $F: D \to D$ be a continuous function and D be a closed, bounded, and convex set, then $\exists p \in D$ s.t. $F(p) = p$ (fixed-point).

We will use BFT to show existence of NE in game (A, B). Essentially we will construct a function whose fixed-points are exactly the NE of (A, B). Then existence of fixed-point implies existence of NE.

Define $F: \Delta_1 \times \Delta_2 \longrightarrow \Delta_1 \times \Delta_2$

$$(x, y) \longrightarrow (x', y')$$

$$\forall i \in S_1: \quad x_i' = \frac{x_i + \sigma_i(x,y)}{\sum_k x_k + \sigma_k(x,y)}, \text{ where } \sigma_i(x,y) = \max\{0, (Ay)_i - x^T A y\}$$

$$\forall j \in S_2: \quad y_j' = \frac{y_j + \tau_j(x,y)}{\sum_k y_k + \tau_k(x,y)}, \text{ where } \tau_j(x,y) = \max\{0, (x^T B)_j - x^T B y\}$$

Lemma 3: If (x, y) NE then $x' = x$, $y' = y$, i.e. (x,y) is a fixed point

pf: It suffices to show that

1. $\forall i \in S_1: \ \sigma_i(x, y) = 0$
2. $\forall j \in S_2: \ \tau_j(x, y) = 0$

For ①, note that $\forall i \in S_1$,

$$(Ay)_i \leq \max_k (Ay)_k \underset{\uparrow}{=} x^T A y \Rightarrow \sigma_i(x,y) = 0$$

$(\because (x,y)$ is NE$)$

Similarly $\forall j \in S_2$

$$(x^T B)_j \leq \max_k (x^T B)_k = x^T B y \Rightarrow \tau_j(x,y) = 0$$

∎

<u>Lemma 4:</u> If (x,y) is a fixed-point, i.e. $x' = x$, $y' = y$, then it is a NE.

<u>Proof:</u> Home Work!!

☆ Scale-Invariance:

<u>Lemma 5:</u> If (x,y) is a NE of game (A,B) then

① it is also a NE of game $(\alpha A, \beta B)$ where $\alpha, \beta > 0$

② it is also a NE of game $(A+\alpha, B+\beta)$ for any $\alpha, \beta \in \mathbb{R}$.

<u>Proof:</u> Follows from the characterization in Theorem 1. ∎

☆ Symmetric Games:

... is symmetric if

☆ Symmetric Games :

A game is said to be symmetric if $S_1 = S_2 = S$ & $B = A^T$. Essentially the two players are indistinguishable. Because

$$P1\text{'s payoff from } (i,j) = A_{ij} = B_{ji} = P2\text{'s payoff from } (j,i)$$

$$P2\text{'s payoff from } (i,j) = B_{ij} = A_{ji} = P1\text{'s payoff from } (j,i)$$

Nash '51 : $\exists$ a symmetric NE. NE (x,y) where $y = x$.

The proof is similar to that for the existence of NE in general game.

* Characterization:

(x, x) is a symmetric NE of game (A, A^T)

iff

$$\forall i \in S : x_i > 0 \Rightarrow (Ax)_i = \max_k (Ax)_k$$

☆ Reduction : Game $\longrightarrow$ Symmetric Game

Let (A, B) be a given game.

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Define $A' = A + \min_{i,j} A_{ij} + 1$

$B' = B + \min_{i,j} B_{ij} + 1$.

Clearly $A' > 0$, $B' > 0$.

$\rightarrow$ By lemma 5 $NE(A,B) = NE(A',B')$.

Construct symmetric game (C, C^T), with strategy set, $S = S_1 \cup S_2$, where
of both players being

$$C = \begin{array}{c} 1 \\ \vdots \\ m \\ m+1 \\ \vdots \\ m+n \end{array} \begin{bmatrix} 0 & A' \\ B'^T & 0 \end{bmatrix} \quad \overset{z}{\left[\begin{array}{c} x \\ \hline y \end{array}\right]}$$

Theorem 2: $z = (x \mid y)$ is a symmetric NE then $x > 0$, $y > 0$, and

$$\left(\frac{x}{\sum_{i=1}^{m} x_i}, \frac{y}{\sum_{j=1}^{n} y_j} \right)$$ is a NE of game (A', B')

Proof: Homework!

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Steps:

1. $y > 0$, o.w. we get i s.t. $z_i > 0$ & $(Cz)_i < \max_k (Cz)_k$
2. Similarly $x > 0$
3. Given $x > 0$, $y > 0$ show that we get NE of (A', B')