1. (5 points) Consider a two player game where the set of strategies of the first and second players are $S_1$ and $S_2$ respectively. Let $m = |S_1|$ and $n = |S_2|$, then such a game can be represented by two $m \times n$ payoff matrices $(A, B)$. In addition, suppose, $A(i, j), B(i, j) > 0, \forall i \in S_1, \forall j \in S_2$. Consider the symmetric game $(C, C^T)$ with the following $(m + n) \times (m + n)$-dimensional block-matrix:

$$
\begin{bmatrix}
0 & A \\
B^T & 0
\end{bmatrix}
$$

Show that a symmetric Nash equilibrium of game $(C, C^T)$ gives a Nash equilibrium of game $(A, B)$.

2. (5 points) Given a two-player game $(A, B)$ of Problem 1, design an algorithm to find a Nash equilibrium of the game. Your algorithm should terminate in finite time.

**Extra credit.** Given that the game has finitely many Nash equilibria, design a finite time algorithm to enumerate all of it’s Nash equilibria.

3. (5 points) Show that if a mixed-strategy profile $(x, y)$ is a Nash equilibrium of game $(A, B)$, then matrix $P$ where $p_{ij} = x_i \cdot y_j$ is a correlated equilibrium (CE) of the game.
4. (10 points) 1-dimensional Sperner’s is defined on a 1-dimensional grid from \([0, 2^n - 1]\), with each integer being a grid point. There are two colors red and blue represented by 0 and 1 respectively. There is a Boolean circuit named \(\text{Color}\) which outputs color (0/1 bit) of a grid point given its bit representation, such that, \(\text{Color}(0)\)=red, \(\text{Color}(2^n - 1)\)=blue, and the rest gets any color. Show that there exists an integer \(0 \leq k \leq 2^n - 1\) such that \(\text{Color}(k)\)=red and \(\text{Color}(k + 1)\)=blue. Furthermore, we can compute it in \(O(n)\) calls to the Boolean circuit “\(\text{Color}\)”. Finally, show that checking if there are more than one such \(k\)s is NP-hard (hint: reduce from 3-SAT).

5. (5 points) Problem 1.2 of the AGT book.

**Note.** Since finding a Nash equilibrium is PPAD-hard in the worst-case, the next immediate question is to do beyond worst-case analysis. Average case, where the game is sampled uniformly at random, is one such regime. This question sheds light on average-case complexity of Nash equilibrium computation.

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**The remaining problems are for self study. Do NOT submit for grading.**

- Problem 1.3 of the AGT book.
- Show that checking if a two-player game has more than one Nash equilibrium is NP-hard (hint: reduce from the problem of checking if a graph has clique of size \(k\).

- (Open problem) Rank of a two-player game \((A, B)\) as \(\text{rank}(A + B)\). Adsul et al (STOC’11) showed that computing Nash equilibrium of a rank-1 game can be done in polynomial time by reducing the problem to 1-dimensional fixed-point. On the other hand games with rank 2 and more are PPAD-hard.

  Show that checking if a rank-1 game has more than one Nash equilibrium is NP-hard. NP-hardness for even constant rank would be good.

- Problem 1.5 of the AGT book.

- There is a weaker notion than CE called coarse-correlated equilibrium (CCE). Here the mediator announces the joint distribution matrix \(P\), and asks each player to opt in or out before it starts suggesting them. If the player chooses to opt out then it can play whatever it wants, on the other hand if it chooses to opt in then it has to play what the mediator suggests. In other words, a player can not get the suggestions and then not play what is suggested. Matrix \(P\) is called CCE of game \((A, B)\) if no player wants to opt out if everyone else is opting in.

  (a) Show that every correlated equilibrium is a coarse-correlated equilibrium.

  (b) Show that all the coarse-correlated equilibria of game \((A, B)\) can be captured by a linear feasibility problem formulation.