1. (5 points) Consider a two player game where pure-action sets of player one and two are respectively $S_1$ and $S_2$. Let $m = |S_1|$ and $n = |S_2|$. Then such a game can be represented by two $m \times n$ dimensional matrices $(A, B)$. Here, player 1 plays $i \in S_1$ and 2 plays $j \in S_2$ then their respective payoffs are $A(i, j)$ and $B(i, j)$. Let $\Delta_1$ and $\Delta_2$ be the set of probability distributions over $S_1$ and $S_2$ respectively (sets of mixed-strategies of the players). Consider the following function $f : \Delta_1 \times \Delta_2 \rightarrow \Delta_1 \times \Delta_2$ defined by Nash, where $(x', y') = f(x, y)$:

$$
\forall i \in S_1 : \quad x'_i = \frac{x_i + \sigma_i(x, y)}{\sum_{k \in S_1} x_k + \sigma_k(x, y)} \quad \text{where} \quad \sigma_i(x, y) = \max\{0, (Ay)_i - x^T Ay\}
$$

$$
\forall j \in S_2 : \quad y'_j = \frac{y_j + \tau_j(x, y)}{\sum_{k \in S_2} y_k + \tau_k(x, y)} \quad \text{where} \quad \tau_j(x, y) = \max\{0, (x^T B)_j - x^T By\}
$$

Show that if $(x, y)$ is a fixed-point of $f$, i.e., $x' = x$ and $y' = y$, then $\forall i \in S_1, \sigma_i(x, y) = 0$ and $\forall j \in S_2, \tau_j(x, y) = 0$, and in turn $(x, y)$ is a Nash equilibrium of game $(A, B)$.

**Solution.** Since $(x, y)$ is a fixed point of $f$, we have,

$$
\forall i \in S_1 : \quad x_i = \frac{x_i + \sigma_i(x, y)}{\sum_{k \in S_1} x_k + \sigma_k(x, y)}
$$

Let $\sum_{k \in S_1} \sigma_k(x, y) = \alpha$. Note that $\alpha \geq 0$, since, $\sigma_k(x, y) \geq 0$ by definition for all $k$.

Then, using $\sum_{k \in S_1} x_k = 1$, we can write,

$$
\forall i \in S_1 : \quad x_i = \frac{x_i + \sigma_i(x, y)}{1 + \alpha}
$$

$$
\therefore \forall i \in S_1 : \quad \alpha x_i = \sigma_i(x, y) \tag{1}
$$

Now, we want to show that $\forall i \in S_1, \sigma_i(x, y) > 0$. For the sake of contradiction, suppose $\exists i \in S_1$ s.t. $\sigma_i(x, y) > 0$. Equivalently, $\alpha > 0$. Then, using (1), we have,

$$
\alpha x_i > 0 \iff \sigma_i(x, y) > 0
$$

$$
\iff (Ay)_i > x^T Ay \tag{2}
$$

The second equivalence follows from the definition of $\sigma_i(x, y)$. Let $P = \{i \mid x_i > 0\}$. Note that, $\sum_{i \in P} x_i = 1$. Using (2), we also have, $\forall i \in P, (Ay)_i > x^T Ay$. \tag{3}

Now,

$$
x^T Ay = \sum_{i \in S_1} x_i (Ay)_i
$$

$$
= \sum_{i \in P} x_i (Ay)_i + \sum_{i \in S_1 \setminus P} x_i (Ay)_i
$$
\[= \sum_{i \in P} x_i (Ay)_i + \sum_{i \in S_1 \setminus P} 0 (Ay)_i \]
\[= \sum_{i \in P} x_i (Ay)_i \]
\[> \sum_{i \in P} x_i (x^T Ay) \quad \text{(using (3))} \]
\[= (x^T Ay) \sum_{i \in P} x_i \]
\[= (x^T Ay) \]

Thus, we get a contradiction, as required. Hence,

\[
\forall i \in S_1 : \quad \sigma_i (x, y) = 0 \\
\Leftrightarrow \forall i \in S_1 : \quad (Ay)_i \leq x^T Ay \\
\Leftrightarrow x^T Ay \geq \max_{i \in S_1} (Ay)_i
\]

However, \(x^T Ay\) is simply a convex combination of \((Ay)_i\)'s, and so, can be at most \(\max_{i \in S_1} (Ay)_i\). Hence, we have,

\[x^T Ay = \max_{i \in S_1} (Ay)_i\]

A similar analysis for player 2 gives

\[x^T By = \max_{j \in S_2} (x^T B)_j\]

Hence, \((x, y)\) is a Nash Equilibrium by definition.
2. (5 points) Given the two player game \((A, B)\) of Problem 1 where \(A(i, j), B(i, j) > 0, \forall i \in S_1, \forall j \in S_2\), consider the symmetric game \((C, C^T)\) with the following \((m + n) \times (m + n)\)-dimensional block-matrix:

\[
C = \begin{bmatrix}
0 & A \\
B^T & 0
\end{bmatrix}
\]

Show that a symmetric Nash equilibrium of game \((C, C^T)\) gives a Nash equilibrium of game \((A, B)\). (That is, show that a symmetric equilibrium \((z, z)\) of the symmetric game, can be used to easily obtain an equilibrium \((x, y)\) of the game \((A, B)\).)

**Solution.** We know \((z, z)\) is a NE of the \((C, C^T)\) game. Clearly \(z\) is a \((m + n)\)-dimensional vector. Let \(z = x|y\) where,

\[
\forall i \leq m : \quad x_i = z_i, \quad \text{and} \quad \forall i \leq n : \quad y_j = z_{m+j}
\]

Then, we can write,

\[
Cz = \begin{bmatrix}
A y \\
B^T x
\end{bmatrix}
\]

(1)

Now, we first prove that \(x \neq 0\). For the sake of contradiction, suppose \(x = 0\).

Now, if \(y = 0\), then \(z = 0\), contradicting that \(z\) is a probability vector. So, \(y \neq 0\).

Therefore, \(\exists j \leq n\) s.t. \(y_j > 0\). Equivalently, for such \(j\),

\[
z_{m+j} > 0
\]

(2)

Now, since \(y \neq 0\), and given \(A > 0\), hence, \((Ay)_i > 0\) for all \(i \leq m\).

Consequently, using (1) we get, \(\max_{l \leq m+n} (Cz)_l > 0\). Further,

\[
(Cz)_{m+j} = (B^Tx)_{j} = 0
\]

\[
\therefore (Cz)_{m+j} < \max_{l \leq m+n} (Cz)_l
\]

(3)

(2) and (3) imply that \(\exists k\) s.t. \(z_k > 0\) \& \((Cz)_k < \max_{l \leq m+n} (Cz)_l\) which contradicts the Nash Equilibrium characterization for \((z, z)\).

Hence, this proves \(x \neq 0\). Similarly, we can show \(y \neq 0\).

Now, let \(\sum_{i \leq m} x_i = \alpha\) and \(\sum_{j \leq n} y_j = \beta\). Clearly, \(\alpha, \beta > 0\) since \(x, y \neq 0\).

Further, let \(x', y'\) be given by \(x'_i = x_i/\alpha \forall i \leq m\) and \(y'_j = y_j/\beta \forall j \leq n\). We now show that \((x', y')\) is a NE of the \((A, B)\) game.

Since \((z, z)\) is a NE of the \((C, C^T)\) game,

\[
\forall k \leq m + n : z_k > 0 \Rightarrow (Cz)_k = \max_{l \leq m+n} (Cz)_l \geq \max_{l \leq m} (Cz)_l
\]

(4)
However, by definition of max, \( \forall k \leq m : (Cz)_k \leq \max_{l \leq m} (Cz)_l \). Hence, using (4), we get,

\[
\forall k \leq m : z_k > 0 \Rightarrow (Cz)_k = \max_{l \leq m} (Cz)_l
\]

\[\therefore \quad \forall i \leq m : x_i > 0 \Rightarrow (Ay)_i = \max_{k \leq m} (Ay)_k\]

\[\therefore \quad \forall i \leq m : x_i/\alpha > 0 \Rightarrow (Ay)_i/\beta = \max_{k \leq m} (Ay)_k/\beta\]

\[\therefore \quad \forall i \leq m : x'_i > 0 \Rightarrow (Ay')_i = \max_{k \leq m} (Ay')_k\]  \(5\)

Similarly, we can also show that

\[
\forall j \leq n : y'_j > 0 \Rightarrow (B^T x')_i = \max_{k \leq n} (B^T x')_k
\]  \(6\)

With (5) and (6), and the fact that \( x', y' \) are valid probability vectors, we have proven that \( (x', y') \) is a NE of the \( (A, B) \) game.
3. (5 points) Show that if a mixed-strategy profile \((x, y)\) is a Nash equilibrium of game \((A, B)\), then matrix \(P\) where \(P_{ij} = x_i \cdot y_j\) is a correlated equilibrium (CE) of the game.

**Solution.** By definition, \(P\) is a CE if

\[
\forall i, i' \in S_1 : \sum_{j \in S_2} P_{ij} A_{ij} \geq \sum_{j \in S_2} P_{ij} A_{i'j}
\]

and,

\[
\forall j, j' \in S_2 : \sum_{i \in S_1} P_{ij} A_{ij} \geq \sum_{i \in S_1} P_{ij} A_{ij'}
\]

We first prove the first condition. Here, \(P\) is defined by \(P_{ij} = x_i y_j\) for all \(i, j\).

Let \(T = \{ i \mid x_i = 0 \}\). Consider 2 cases as follows:

- Consider all \(i \in T\). We have,

\[
\forall i \in T, \forall j \in S_2 : P_{ij} = x_i y_j = 0.
\]

\[\therefore \forall i \in T, \forall i' \in S_1 : \sum_{j \in S_2} P_{ij} A_{ij} = 0 \geq \sum_{j \in S_2} P_{ij} A_{i'j} \quad (1)\]

- Consider all \(i \in S_1 \setminus T\). By definition of \(T\),  \(\forall i \in S_1 \setminus T\), \(x_i > 0\). But since \((x, y)\) is a NE of the game, we get,

\[
\forall i \in S_1 \setminus T : (Ay)_i = \max_{k \in S_1} (Ay)_k
\]

\[\therefore \forall i, i' \in S_1 \setminus T : (Ay)_i \geq (Ay)_{i'}
\]

\[
\iff \sum_{j \in S_2} A_{ij} y_j \geq \sum_{j \in S_2} A_{i'j} y_j
\]

\[
\iff x_i \sum_{j \in S_2} A_{ij} y_j \geq x_i \sum_{j \in S_2} A_{i'j} y_j
\]

\[
\iff \sum_{j \in S_2} A_{ij} (x_i y_j) \geq \sum_{j \in S_2} A_{i'j} (x_i y_j)
\]

\[
\iff \sum_{j \in S_2} A_{ij} P_{ij} \geq \sum_{j \in S_2} A_{i'j} P_{ij} \quad (2)
\]

Thus, (1) and (2) together prove the first condition of the CE. The proof for the second condition is similar.
There is a weaker notion than CE called coarse-correlated equilibrium (CCE). Here, the mediator announces the joint distribution matrix $P$, and asks each player to opt in or out before suggesting them any actions. If a player chooses to opt out, then it can play whatever it wants; on the other hand, if it chooses to opt in, then it has to play what the mediator suggests. In other words, unlike CE, a player can not get the suggestions and then choose to not play what is suggested. Matrix $P$ is called CCE of a game $(A, B)$ if no player wants to opt out IF everyone else is opting in.

(a) (5 points) Show that every correlated equilibrium is a coarse-correlated equilibrium.

(b) (5 points) Show that all the coarse-correlated equilibria of game $(A, B)$ can be captured by a linear feasibility problem formulation.

**Solution.** By definition, if $P$ is a CE, then for player 1, we have

$$\forall i, i' \in S_1 : \sum_{j \in S_2} P_{ij} A_{ij} \geq \sum_{j \in S_2} P_{ij} A_{i'j}$$

(1)

By definition, if $P$ is a CCE, then if player 2 is opting in, then the expected payoff for player 1 by opting in is at least as much as the expected payoff by opting out - by opting for any action $i \in S_1$. Formally,

$$\forall i' \in S_1 : \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{ij} \geq \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{i'j}$$

(2)

Now, given that $P$ is a CE, (1) holds for each $i \in S_1$. Hence, summing the inequality over all $i \in S_1$ precisely gives (2) as required. We can similarly prove the CCE condition for player 2 by using the CE condition for player 1. Hence, every CE is also a CCE.

Next, the Linear feasibility formulation for computing a CCE is given by

$$\forall i' \in S_1 : \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{ij} \geq \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{i'j}$$

(3)

$$\forall j' \in S_2 : \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{ij} \geq \sum_{i \in S_1} \sum_{j \in S_2} P_{ij} A_{ij'}$$

(4)

$$\forall i \in S_1, \forall j \in S_2 : P_{ij} \geq 0$$

(5)

$$\sum_{i \in S_1} \sum_{j \in S_2} P_{ij} = 1$$

(6)

This has $mn$ variables $P_{ij}$ which capture the distribution $P$ used by the mediator. The constraints in (5) and (6) ensure that $P$ is indeed a probability distribution. Constraints in (3) and (4) imply that $P$ is a CCE by definition. It’s easy to verify that all the constraints are linear in the said variables.
5. (5 points) Problem 1.2 of the AGT book.

**Solution.** Let’s compute the probability that there is no pure NE. For any \( j \leq n \), let \( b_j \) be an action in \( S_1 \) which maximizes the payoff of \( p_1 \) when \( p_2 \) plays action \( j \), that is \( A_{b_j} = \max_i A_{ij} \). Since the entries of \( A \) are coming u.a.r. from \([0, 1]\) which is an infinite set, so \( b_j \) is unique with probability 1. Now, if \( j \) is not the action which maximizes \( p_1 \)'s payoff when \( p_2 \) plays \( b_j \), i.e. \( B_{b_j} = \max_k A_{b_j k} \), then there is no PNE corresponding to column \( j \). Let this event be represented by an indicator random variable \( X_j \) - which is 1 if there is no PNE corresponding to column \( j \), 0 otherwise. We want to compute the probability of \( X_j = 1 \) for all \( j \). It’s easy to see that \( X_j \) is a Bernoulli r.v. with \( \Pr(X_j = 1) = 1 - \frac{1}{n} \) (Since the maximum entry in row \( b_j \) of matrix \( B \) could be in any column with equal probability and we want it to not be in \( j \) for \( X_j = 1 \)). Consider two such indicator variables, \( X_i \) and \( X_j \). We can compute,

\[
E[X_i] = E[X_j] = 1 - \frac{1}{n}
\]

\[
Var(X_i) = Var(X_j) = \frac{1}{n} \left( 1 - \frac{1}{n} \right)
\]

\[
E[X_i X_j] = Pr(X_i = 1, X_j = 1)
\]

\[
= Pr(b_i = b_j)Pr(X_i = 1, X_j = 1| b_i = b_j) + Pr(b_i \neq b_j)Pr(X_i = 1, X_j = 1| b_i \neq b_j)
\]

\[
= \frac{1}{n} \left( 1 - \frac{2}{n} \right) + \left( 1 - \frac{1}{n} \right) \left( 1 - \frac{1}{n} \right)^2
\]

\[
= (1 - \frac{1}{n})^2 - \frac{1}{n^3}
\]

\[\therefore \text{cov}(X_i, X_j) = E[X_i X_j] - E[X_i]E[X_j]
\]

\[= -\frac{1}{n^3}
\]

\[\therefore \text{corr}(X_i, X_j) = \frac{\text{cov}(X_i, X_j)}{\sqrt{Var(X_i)Var(X_j)}}
\]

\[= \frac{-1}{n^2(1 - \frac{1}{n})}
\]

Thus \( \text{corr}(X_i, X_j) \to 0 \) as \( n \to \infty \). Hence, for a computation involving just the limiting case of \( n \to \infty \), the variables \( X_i, X_j \) can taken to be independent.

Finally, the game has no PNE if \( X_j = 1 \) \( \forall j \in S_2 \). Using the independence, this equals \( \prod_j Pr(X_j = 1) = (1 - \frac{1}{n})^n \), which approaches \( 1/e \) as \( n \to \infty \).

Hence, the probability of the complement event that there is a PNE, is simply \( 1 - 1/e \).