# Zero-Knowledge Proofs

Lecture 18

# Interactive Proofs

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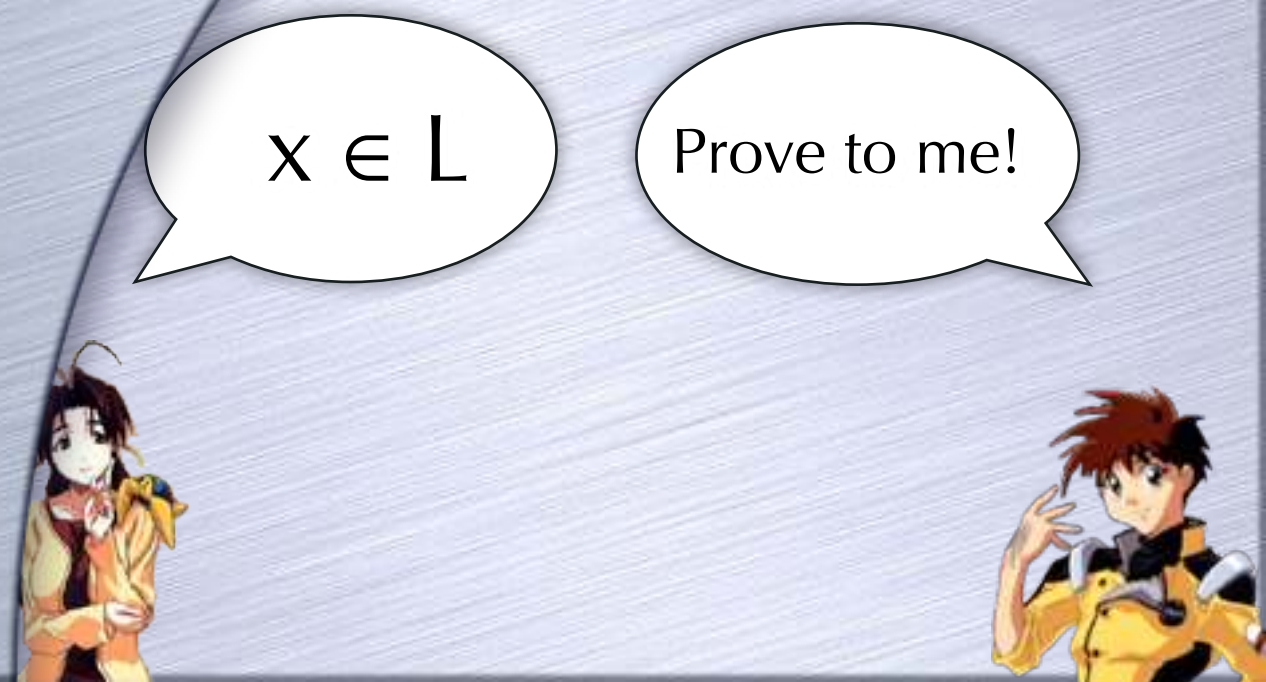

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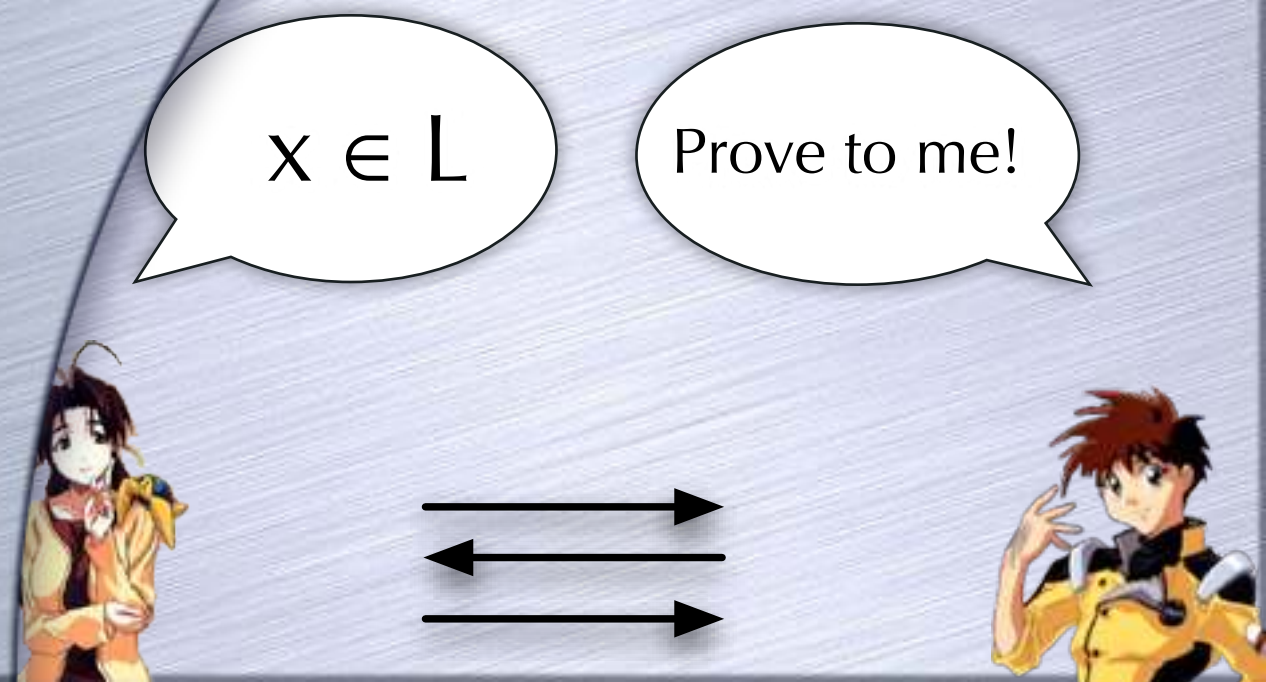

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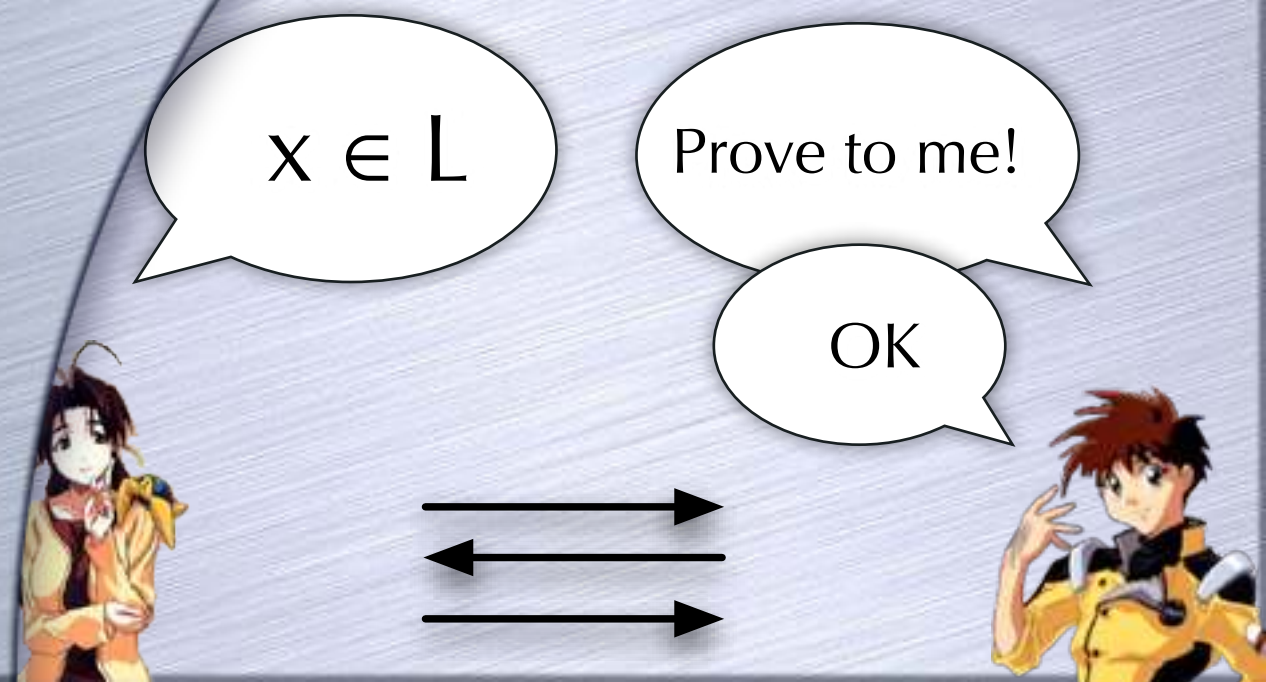

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- ***Prover*** wants to convince ***verifier*** that x has some property
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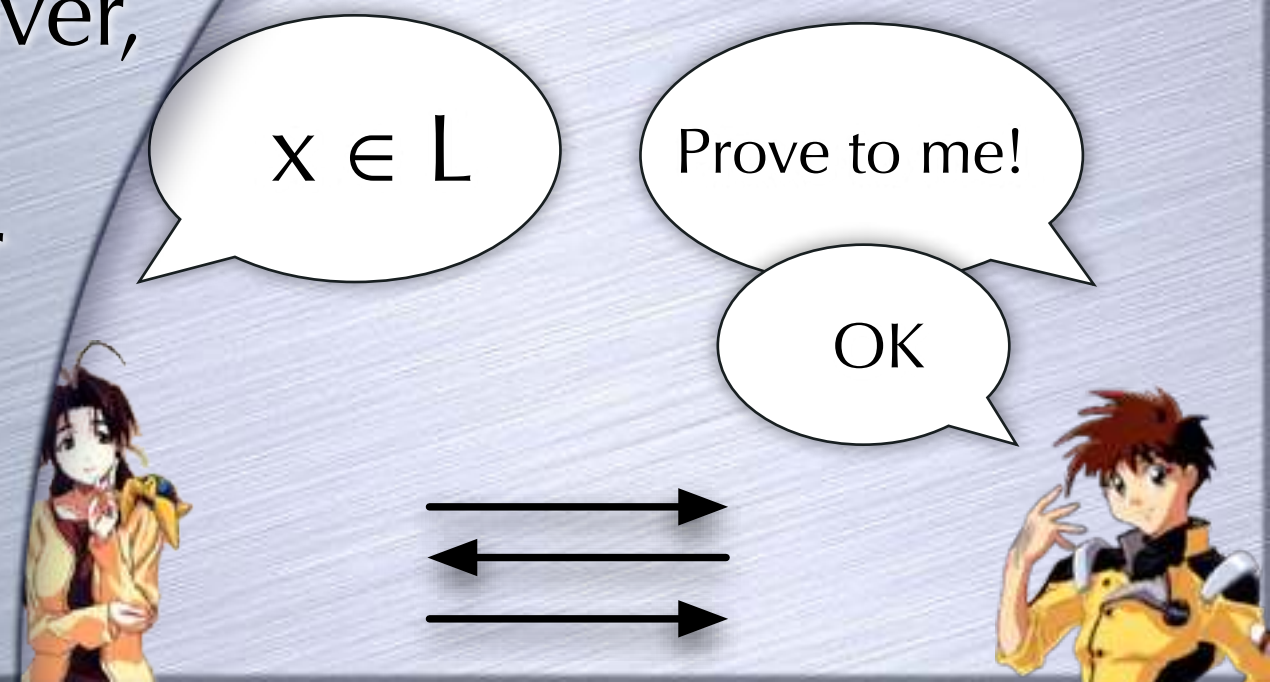
- All powerful prover, computationally bounded verifier (for now)



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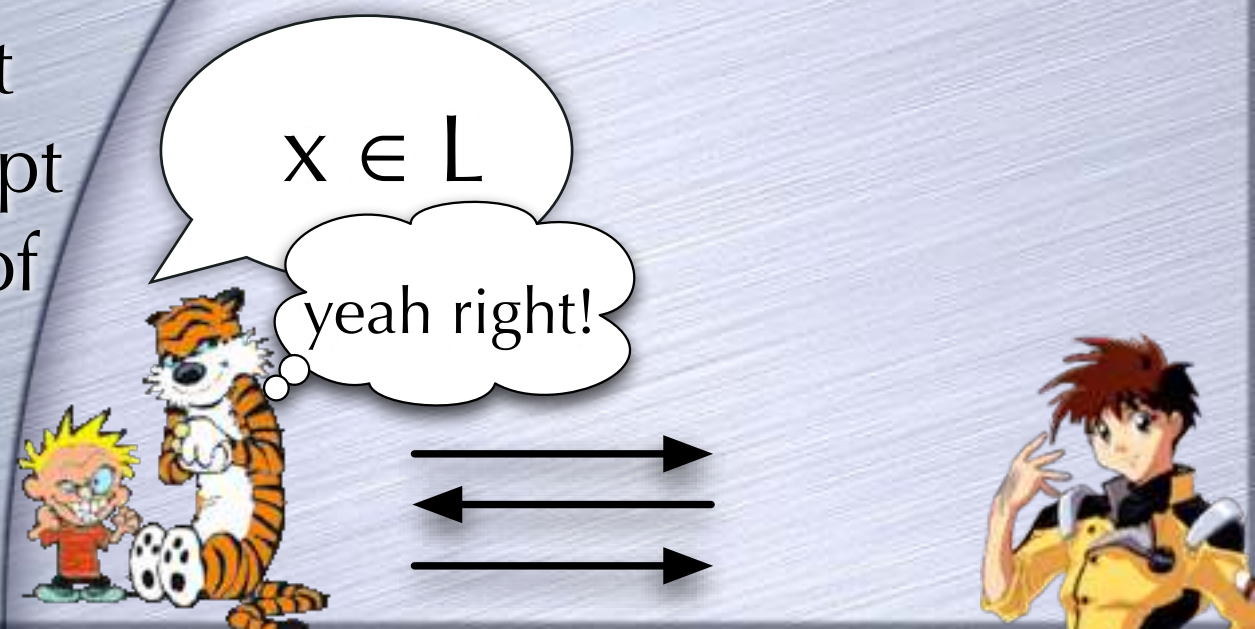

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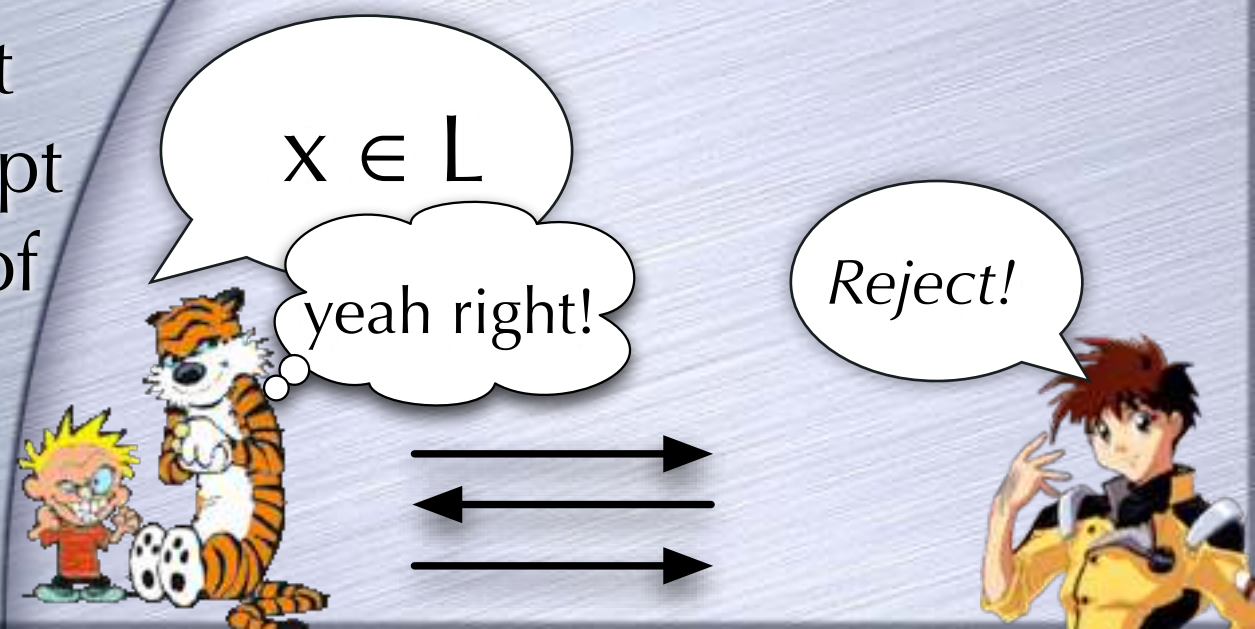

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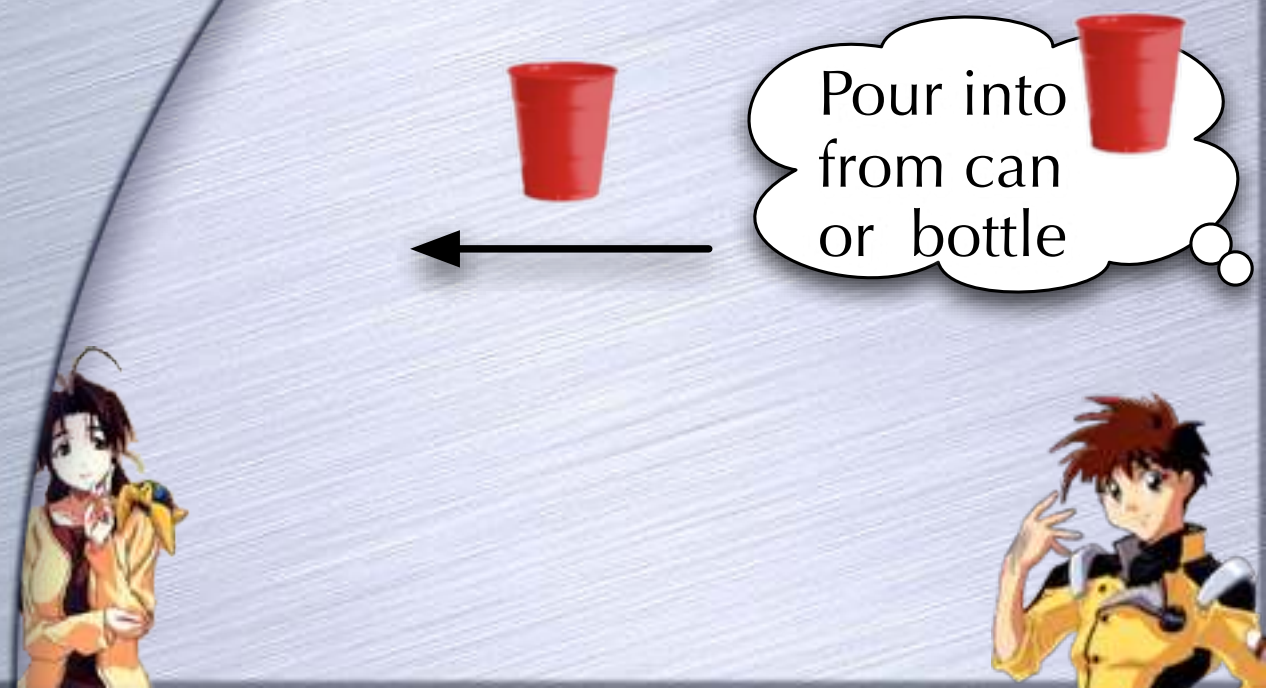

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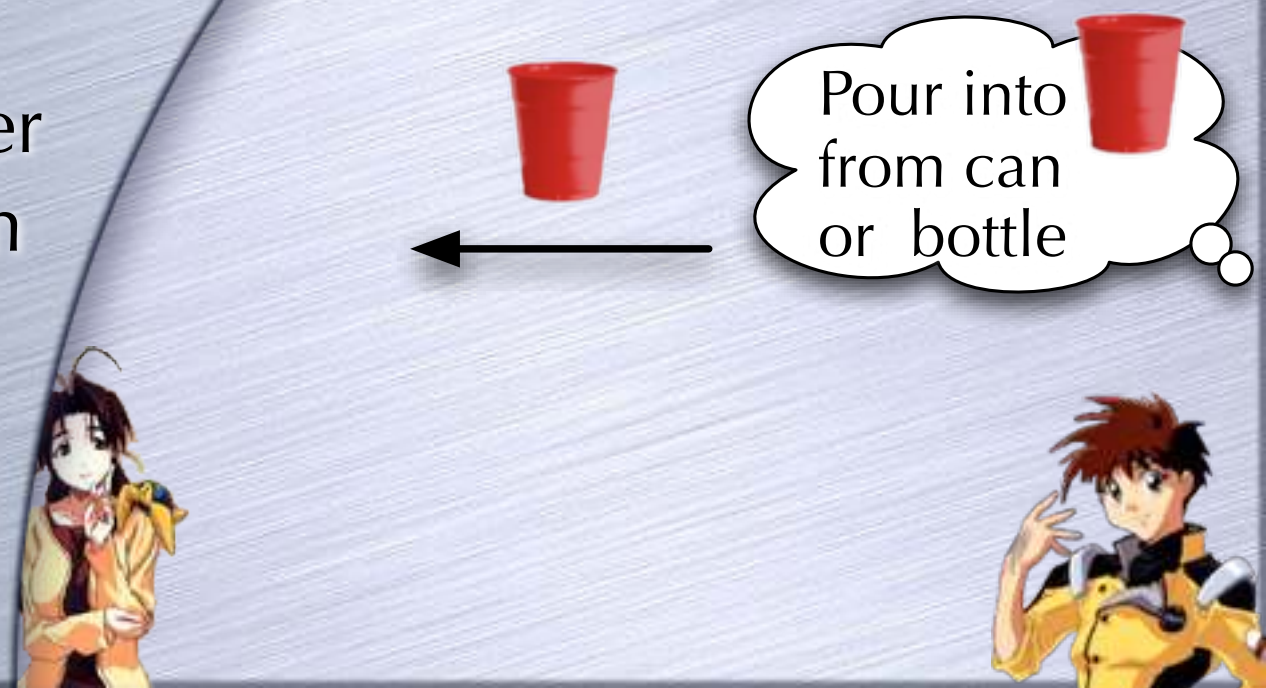

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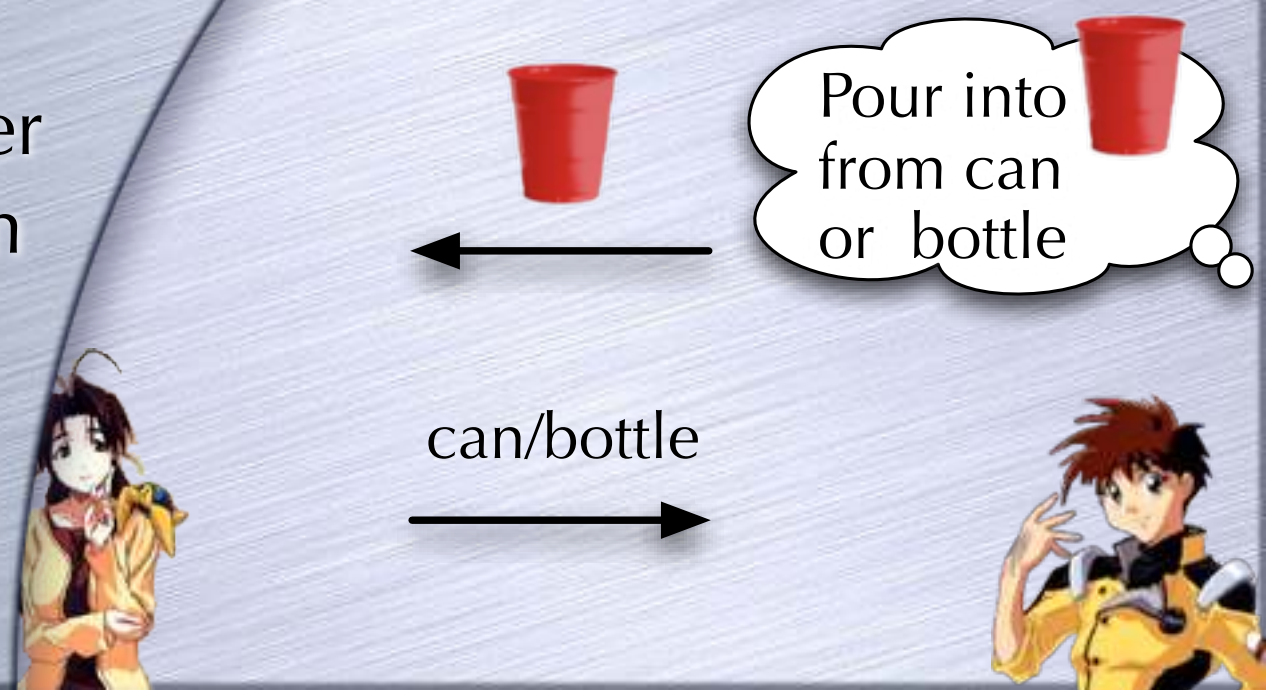

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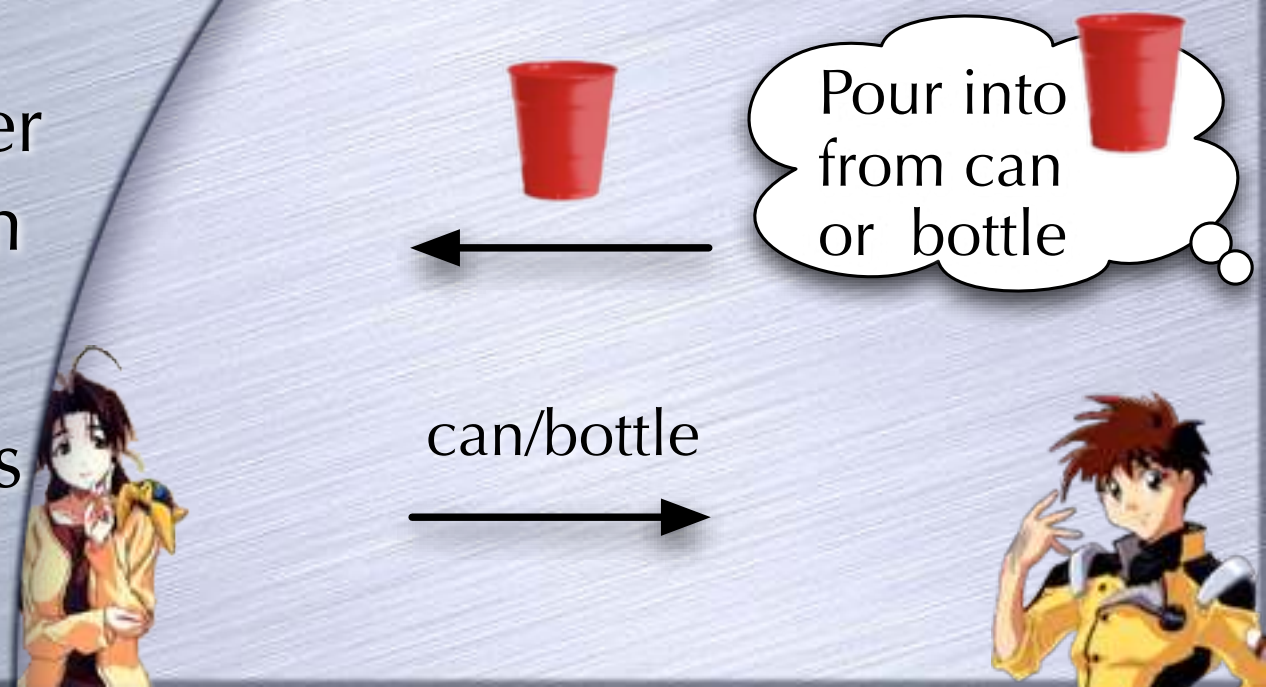

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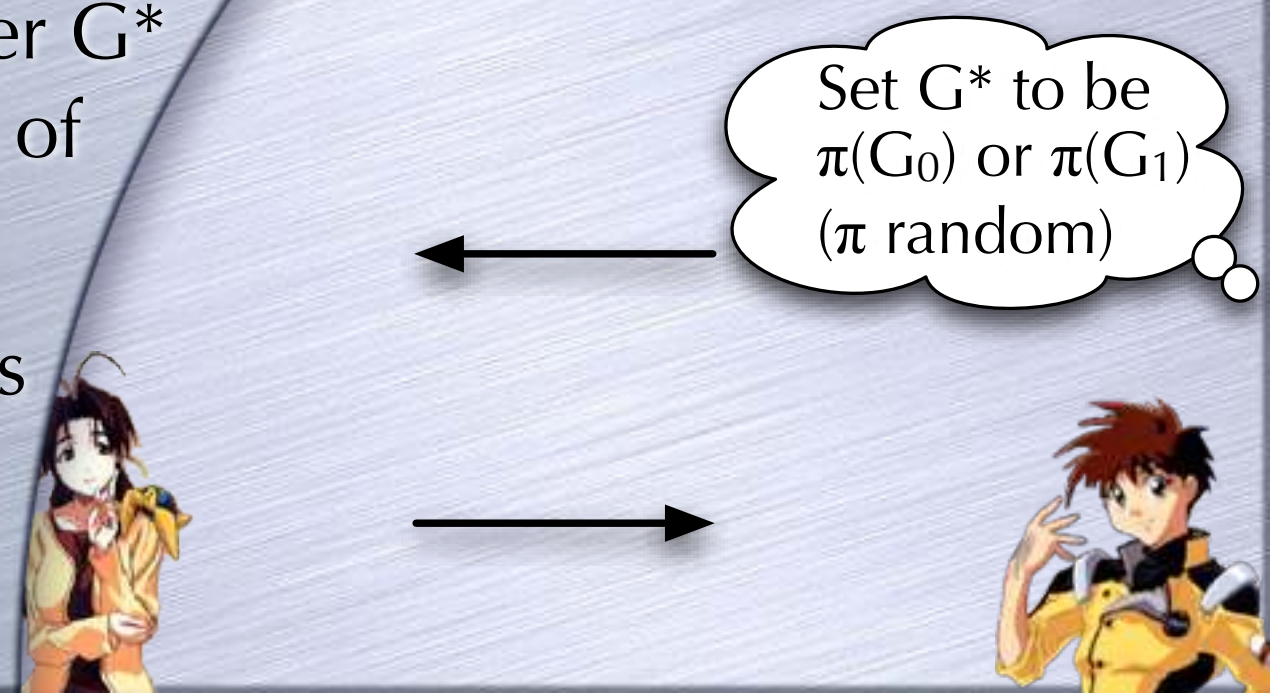

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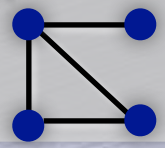
both are isomorphic to the graph represented by the drawing



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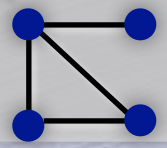

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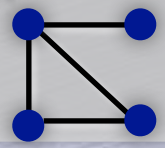

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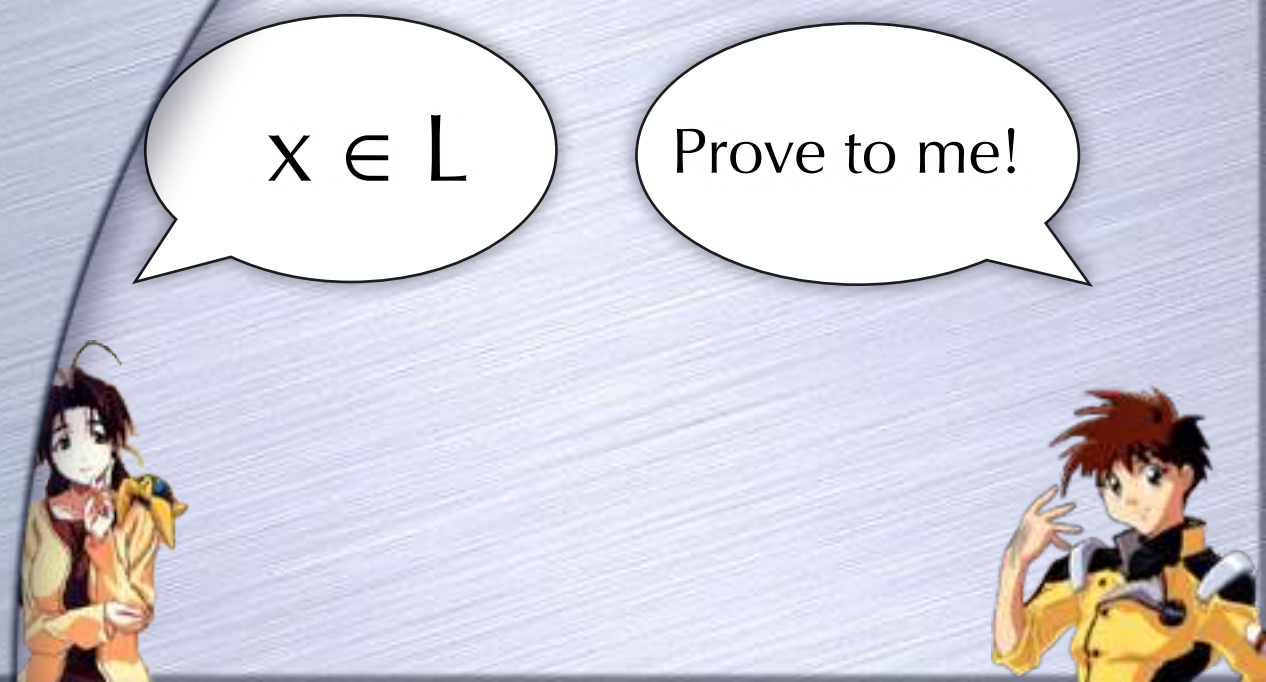

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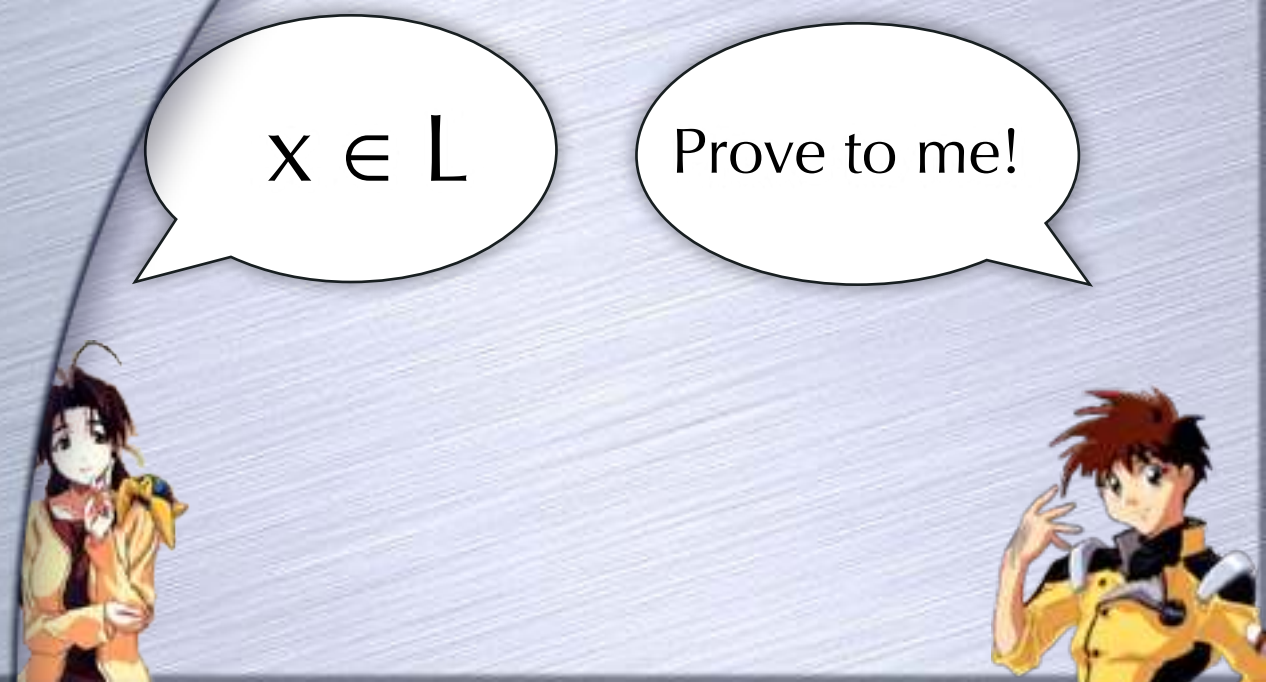

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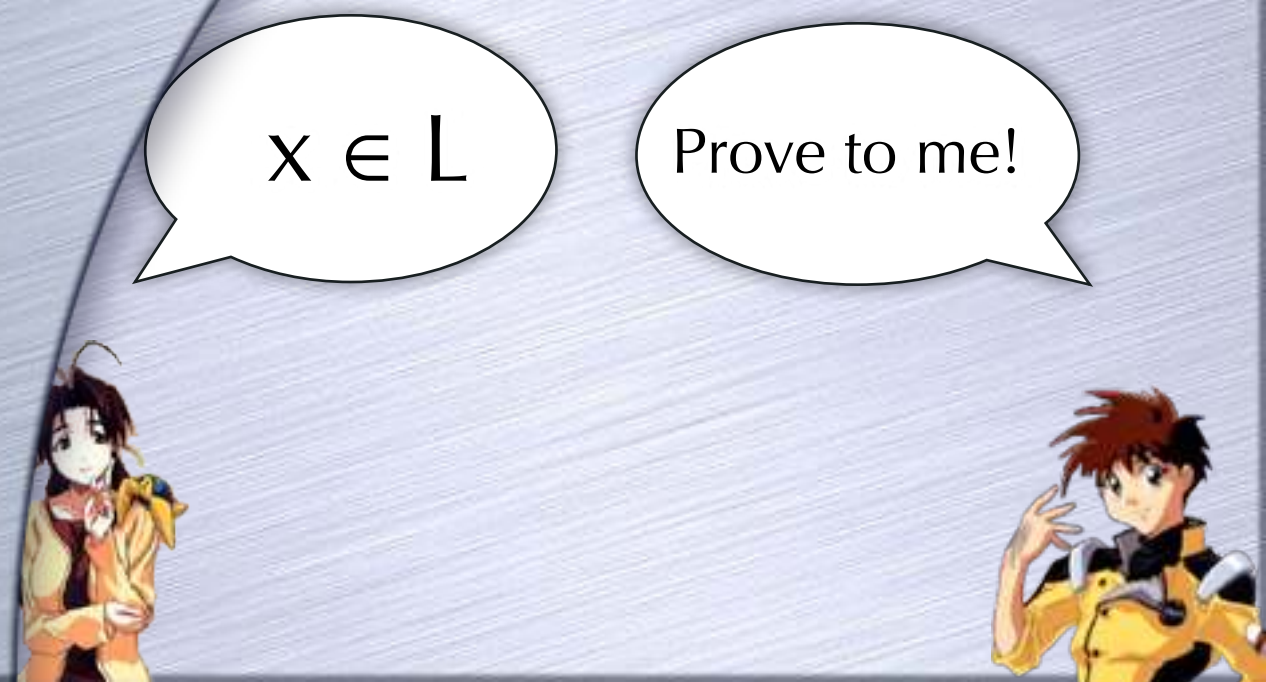

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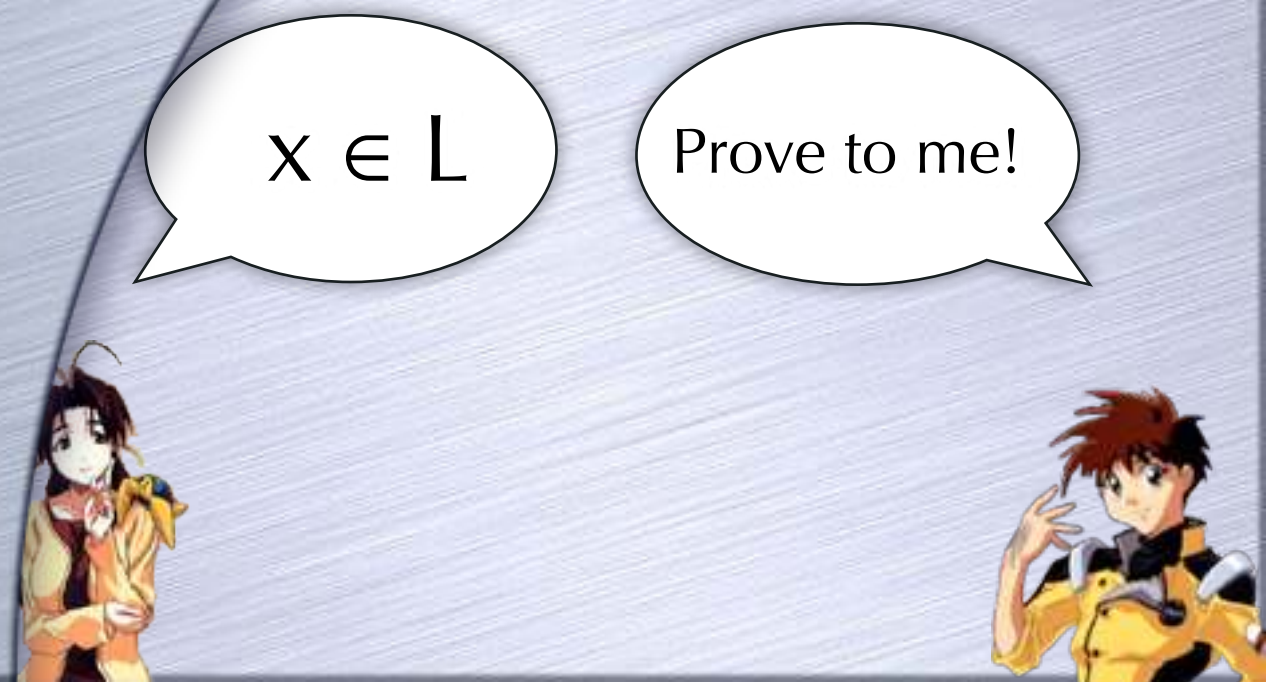

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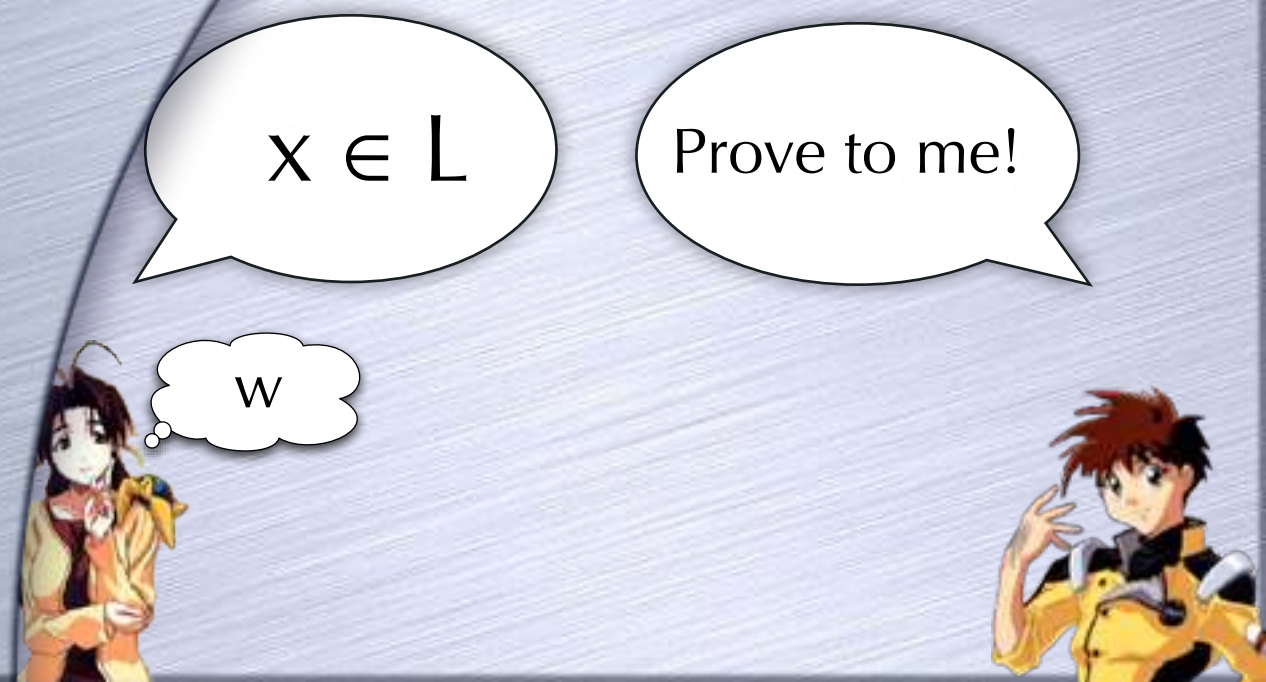

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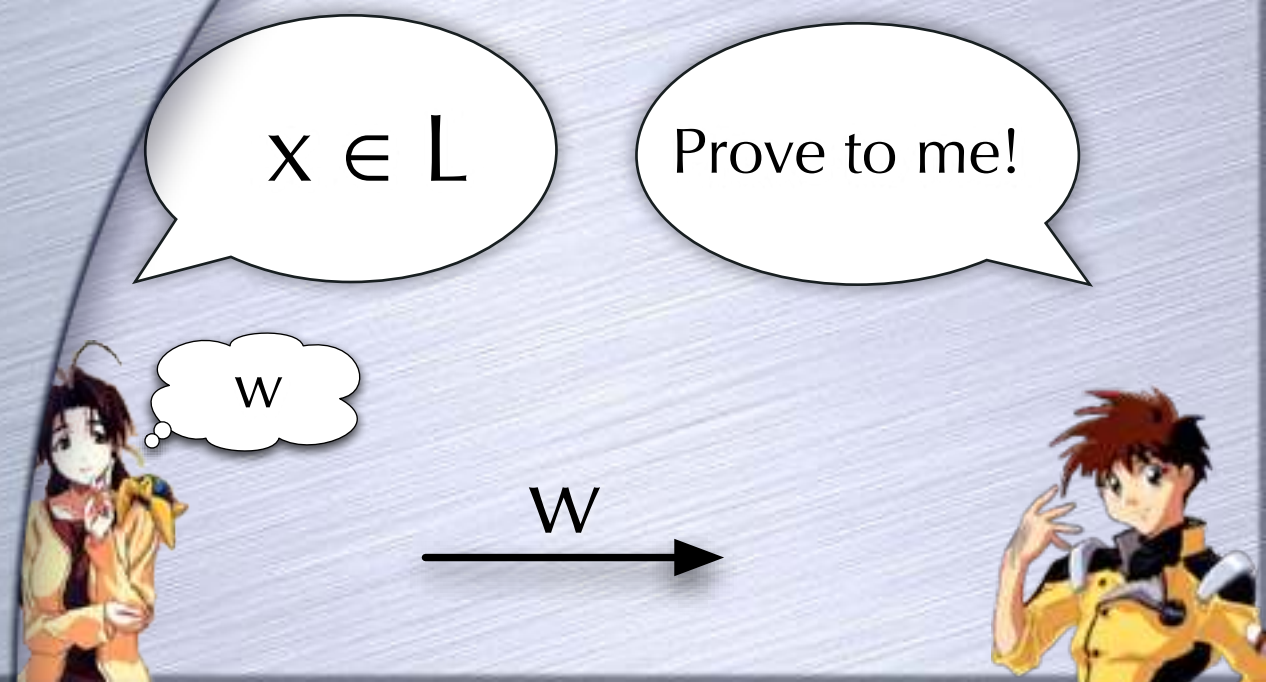

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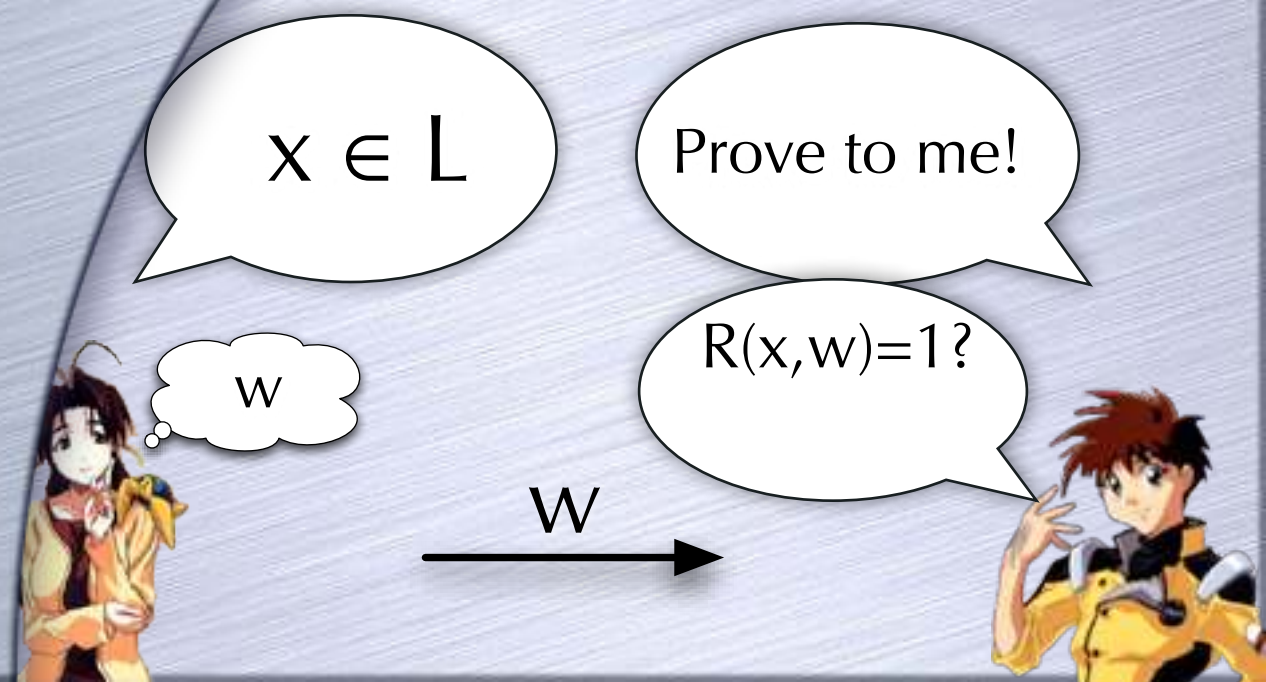

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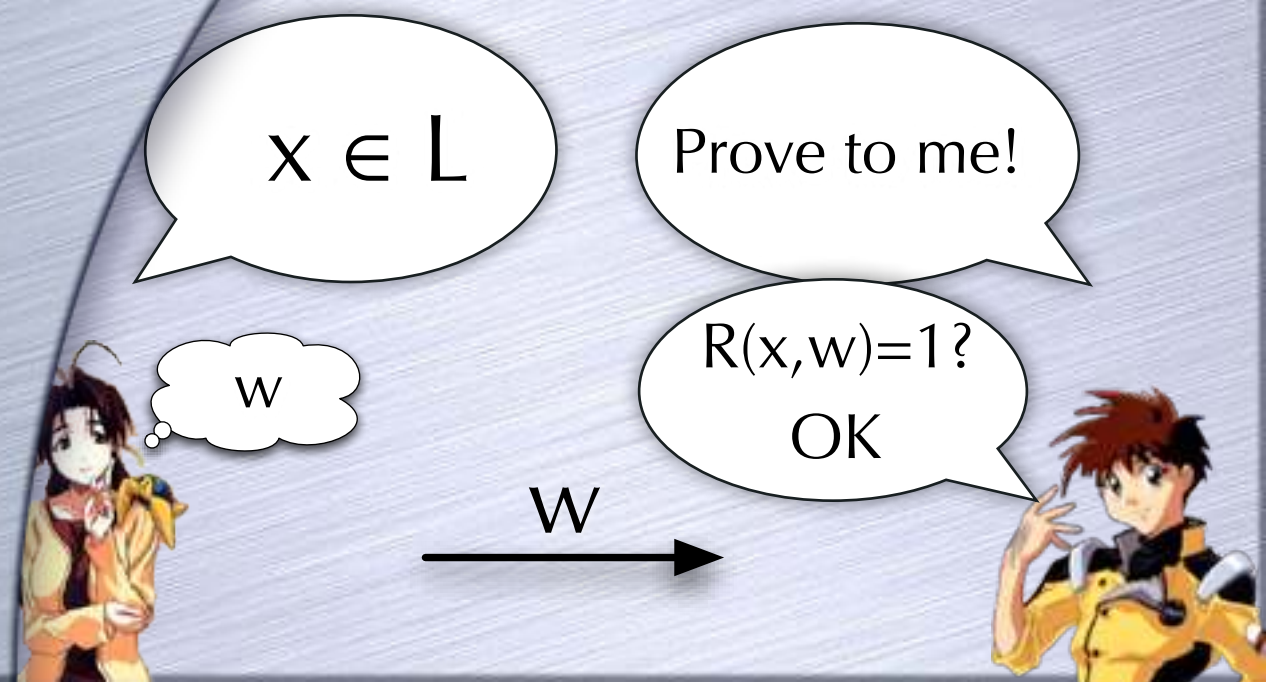

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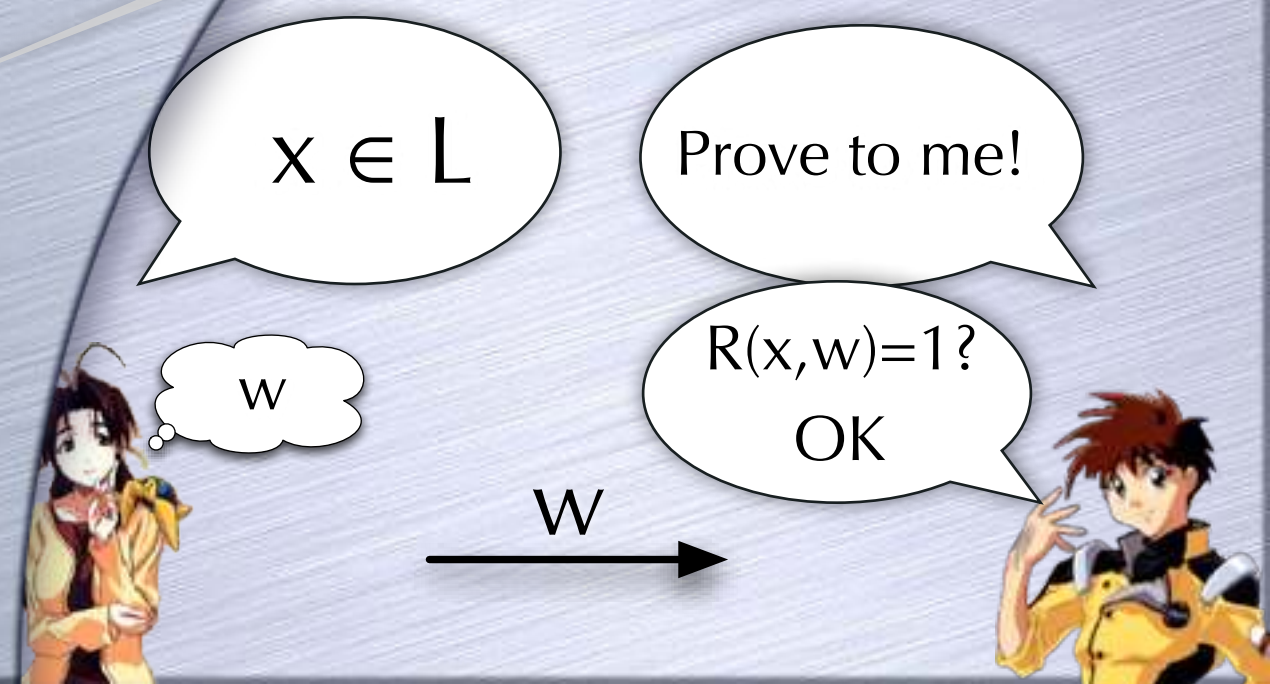

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- *What if prover doesn't want to reveal w?*

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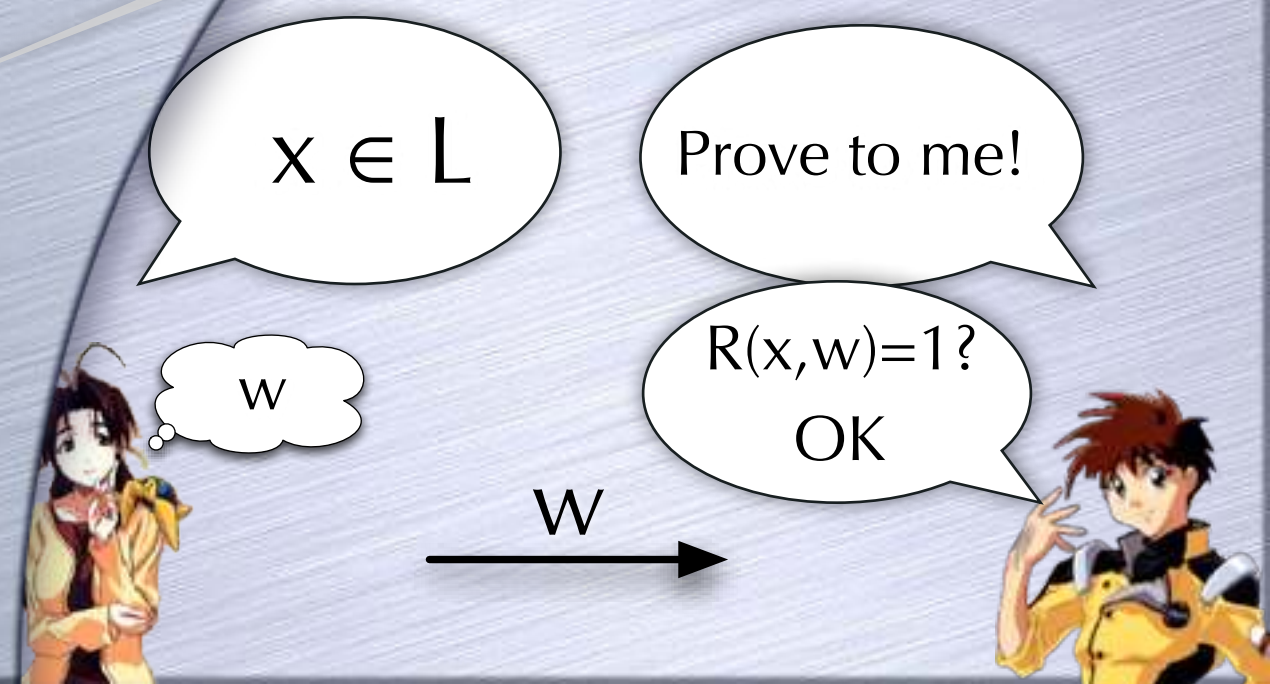

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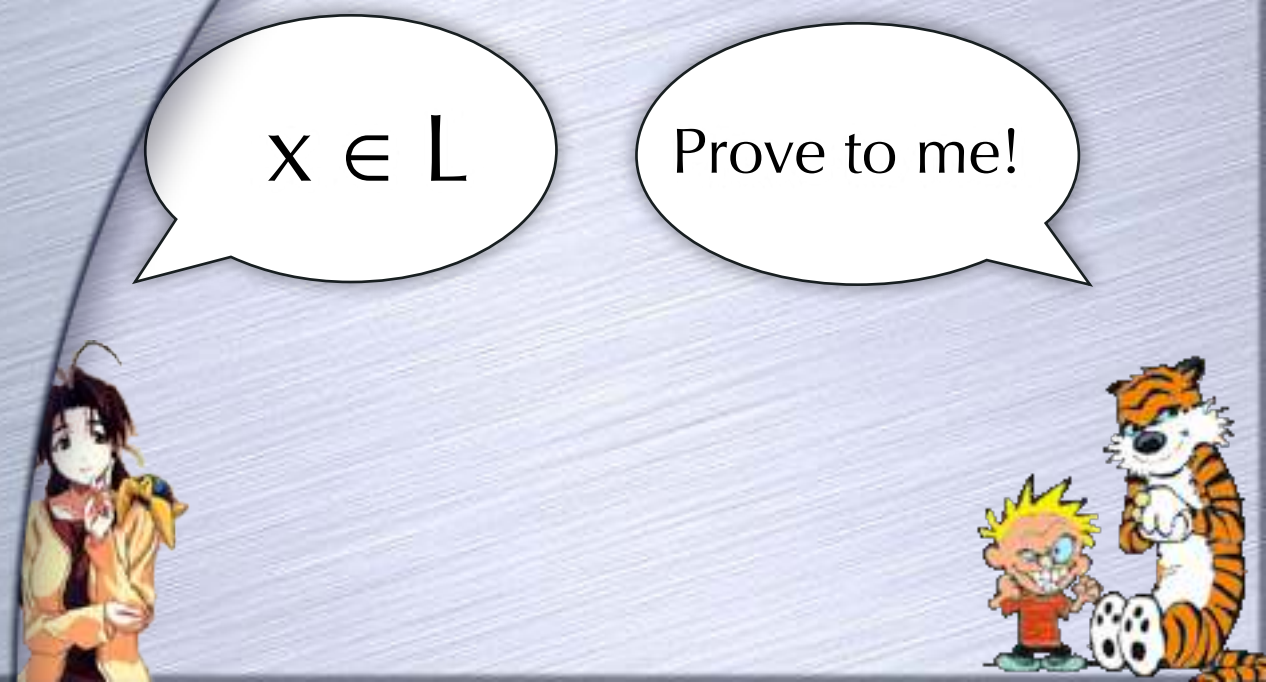

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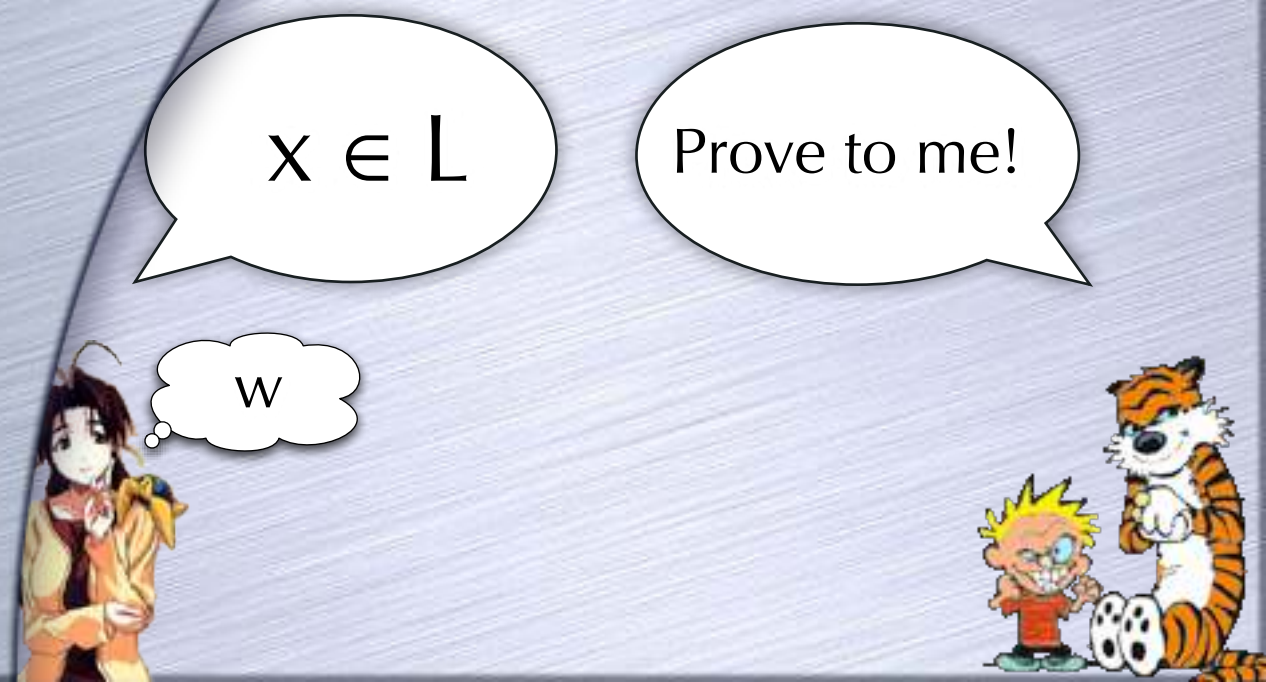

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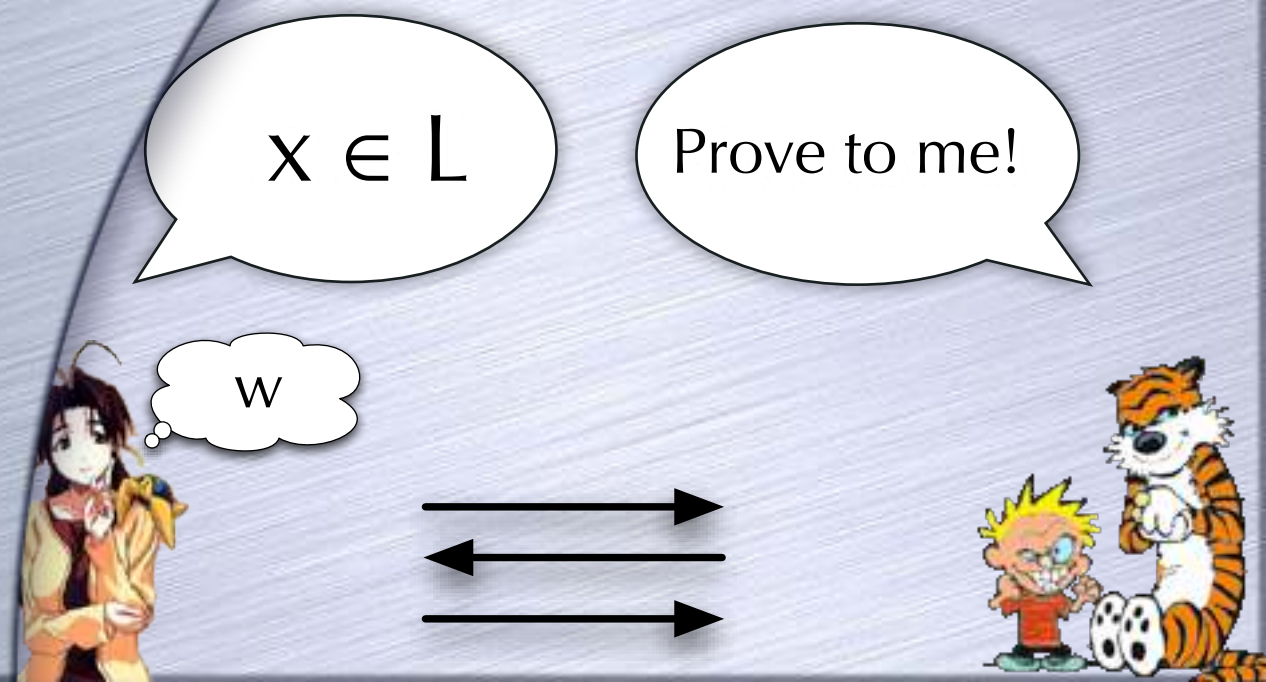

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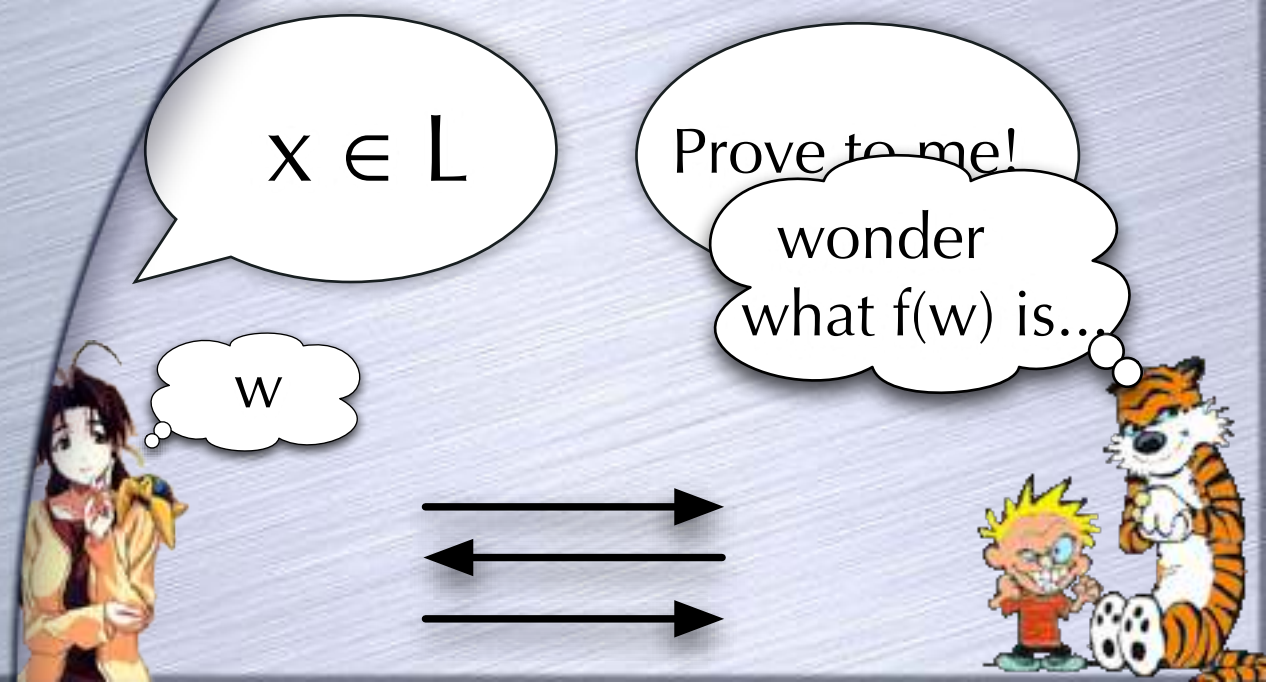

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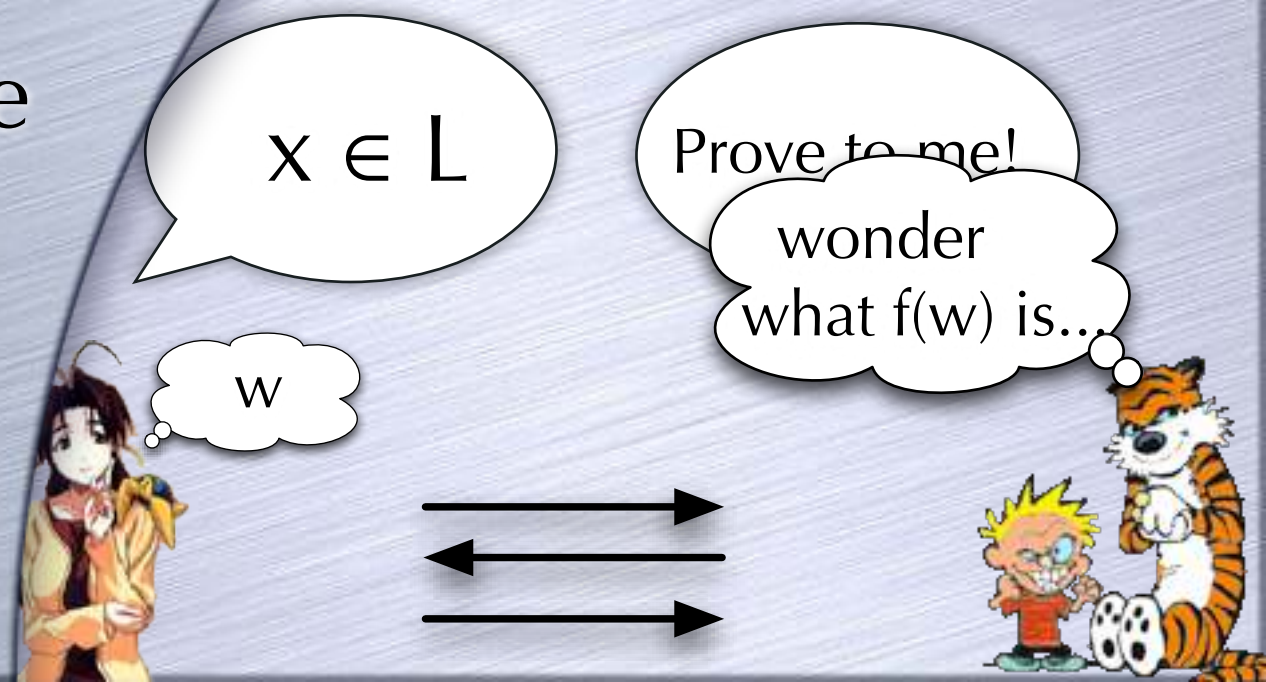

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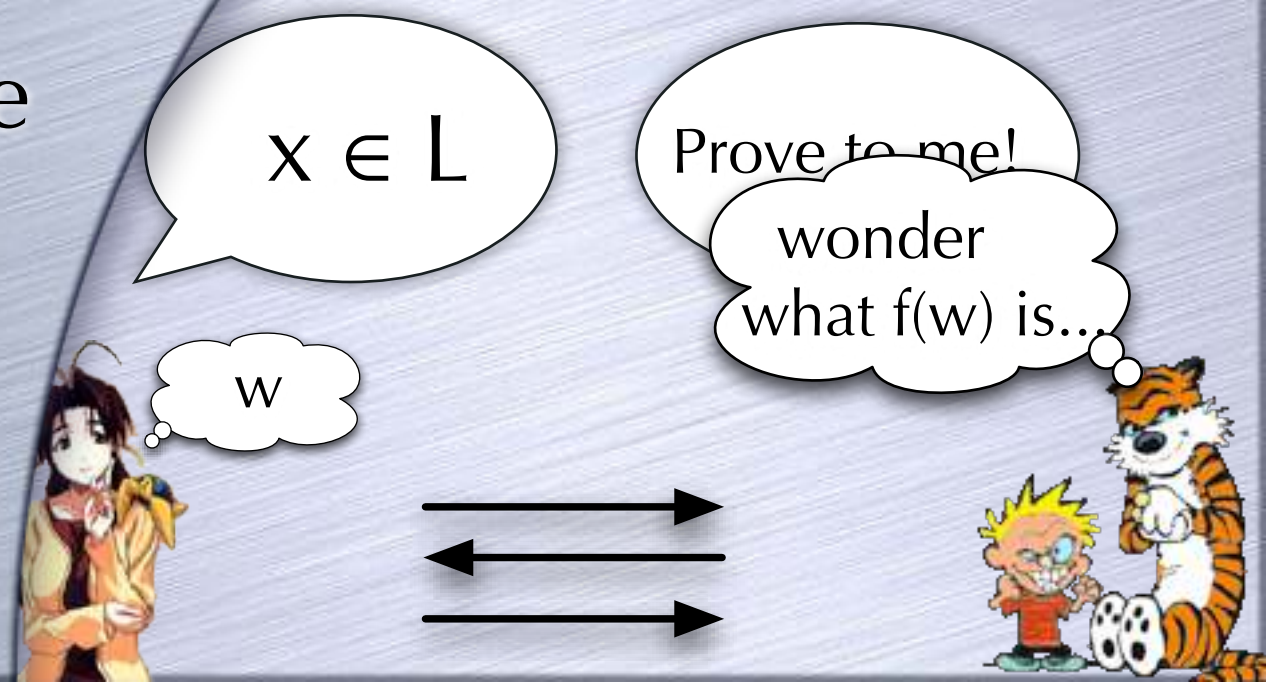

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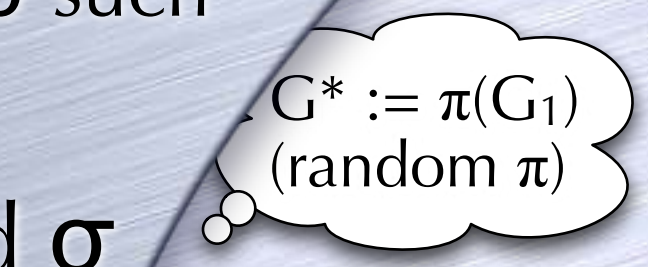


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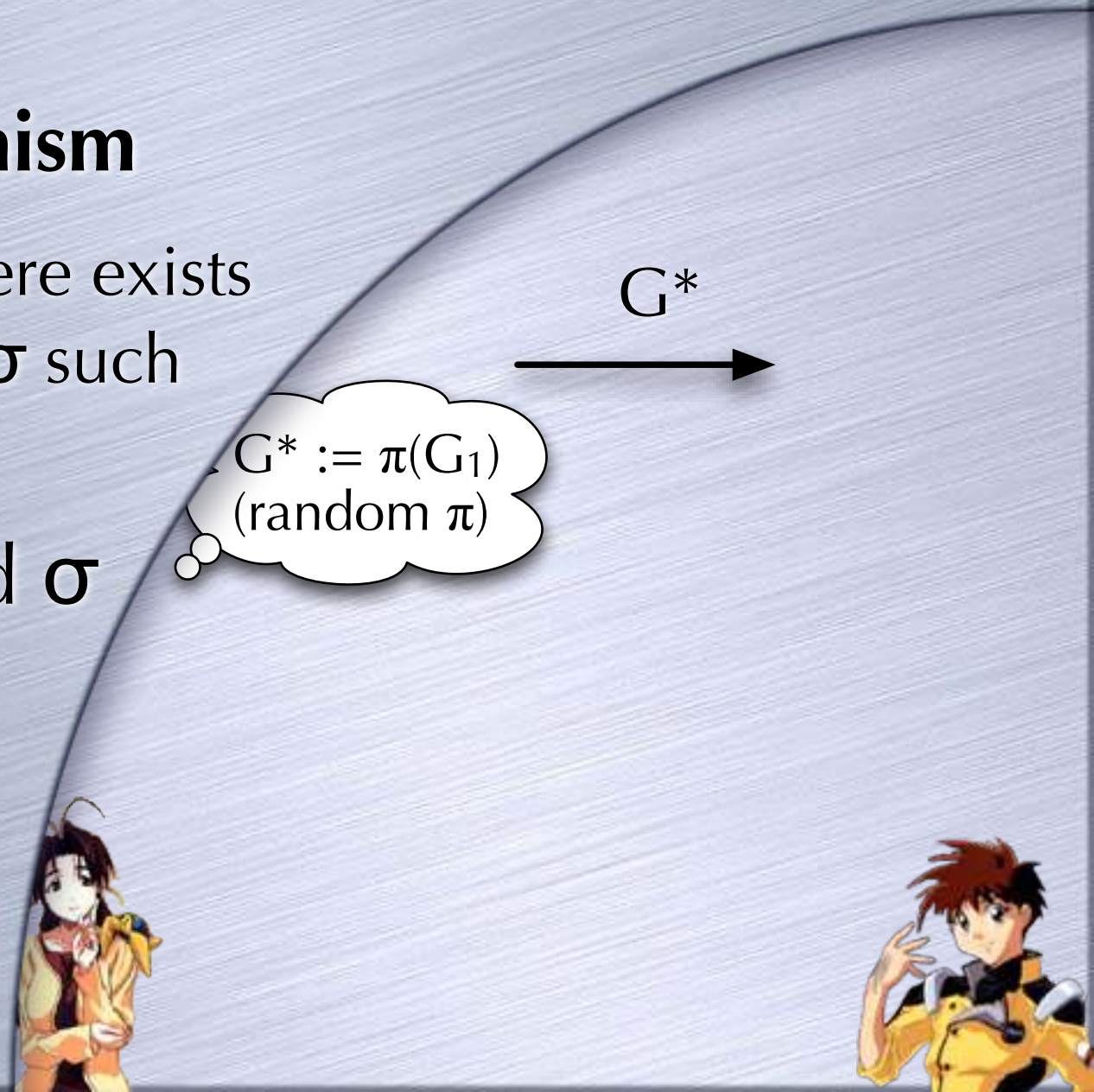

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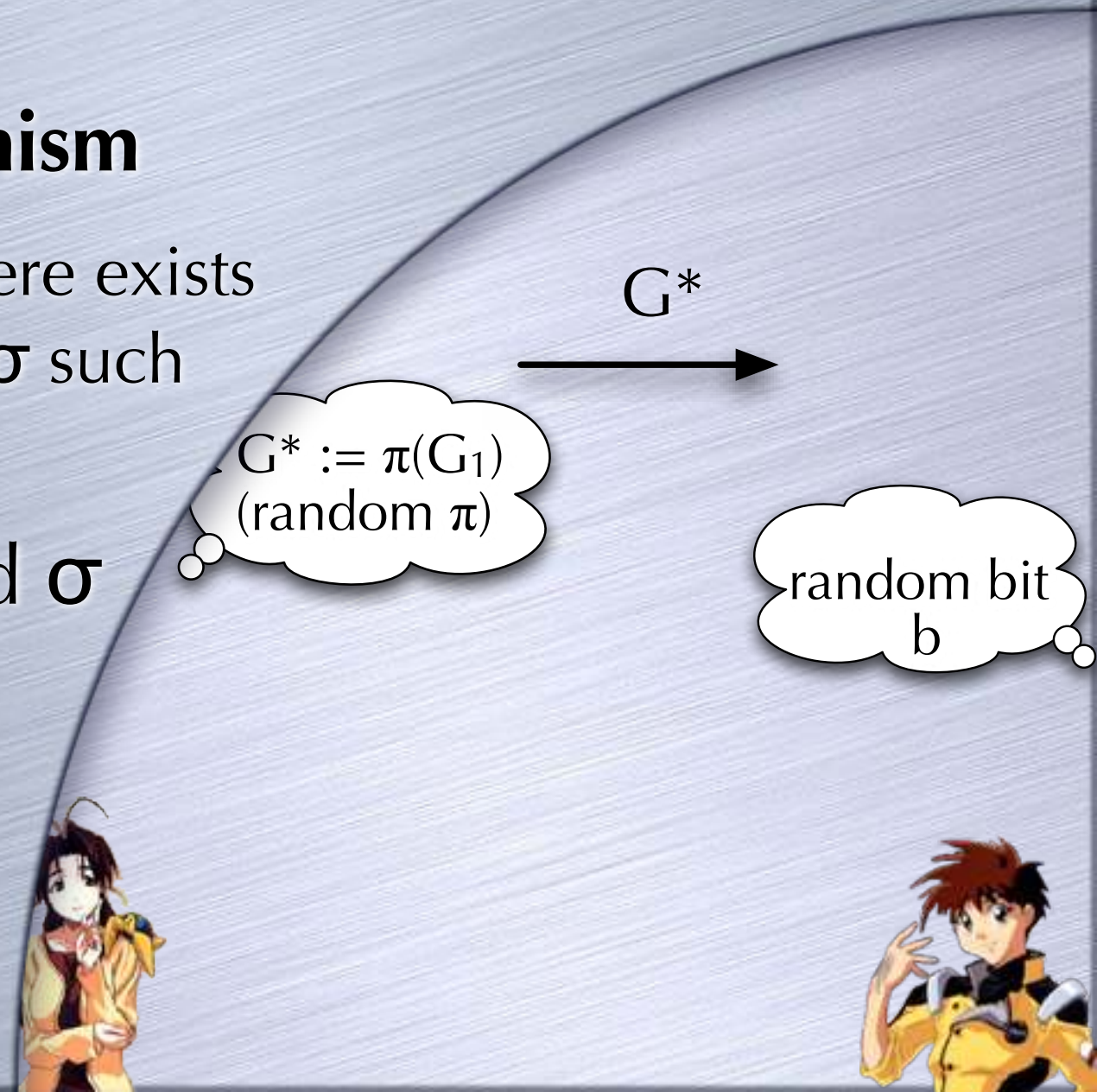

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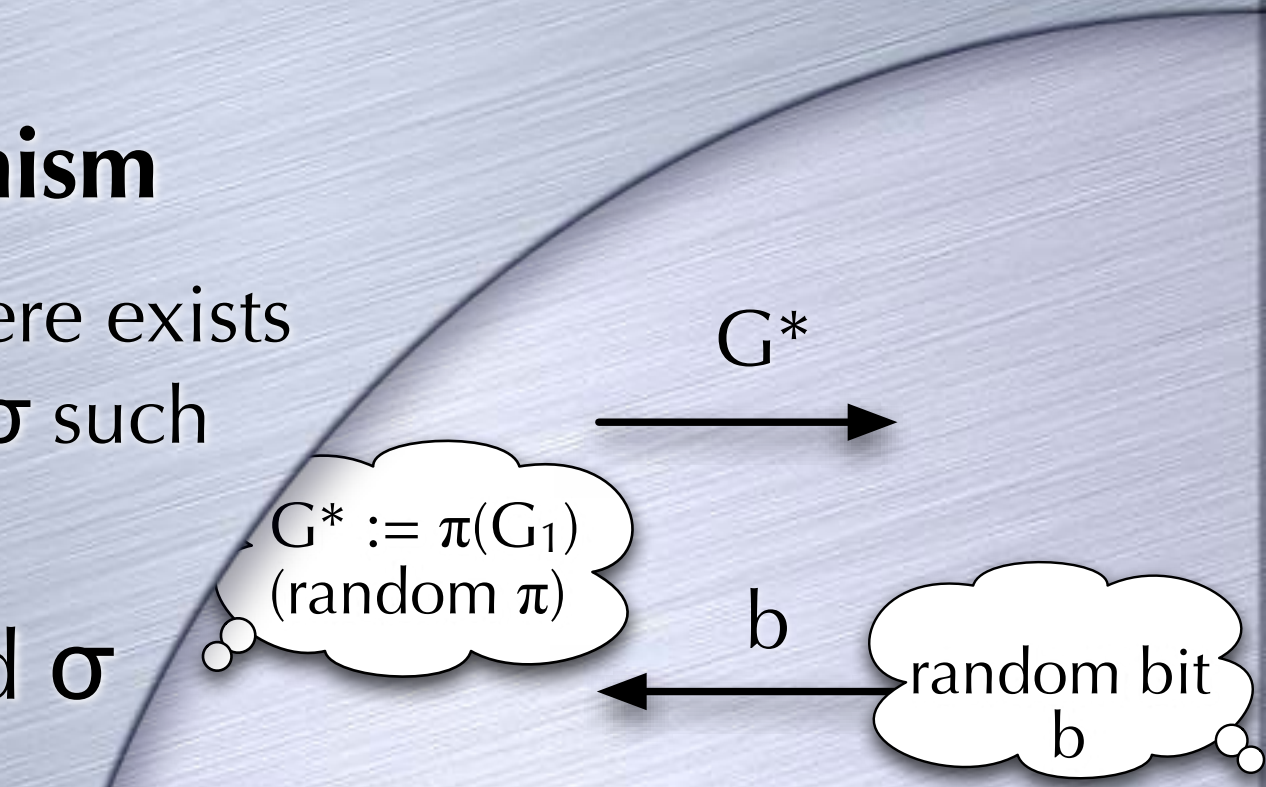


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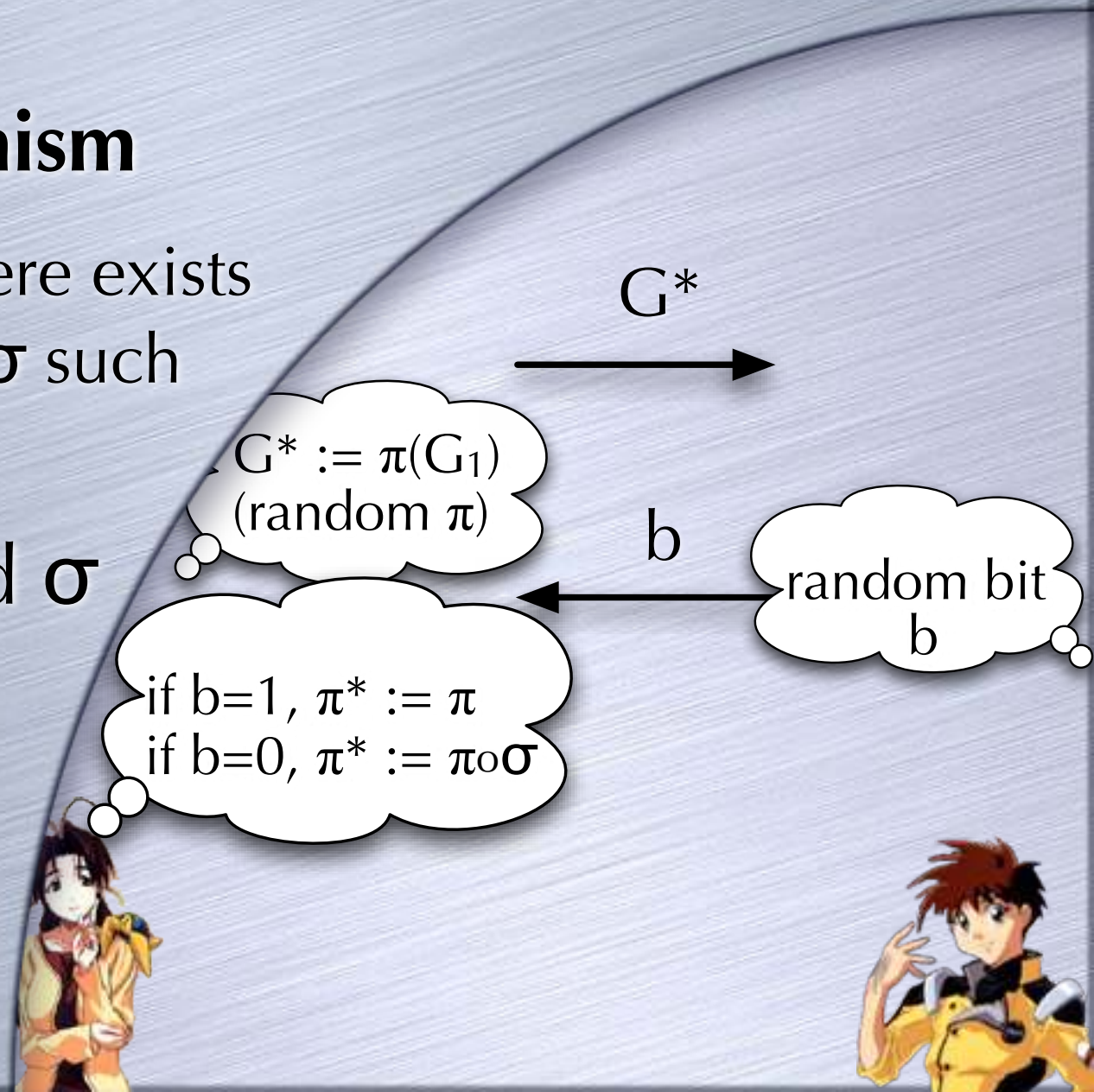

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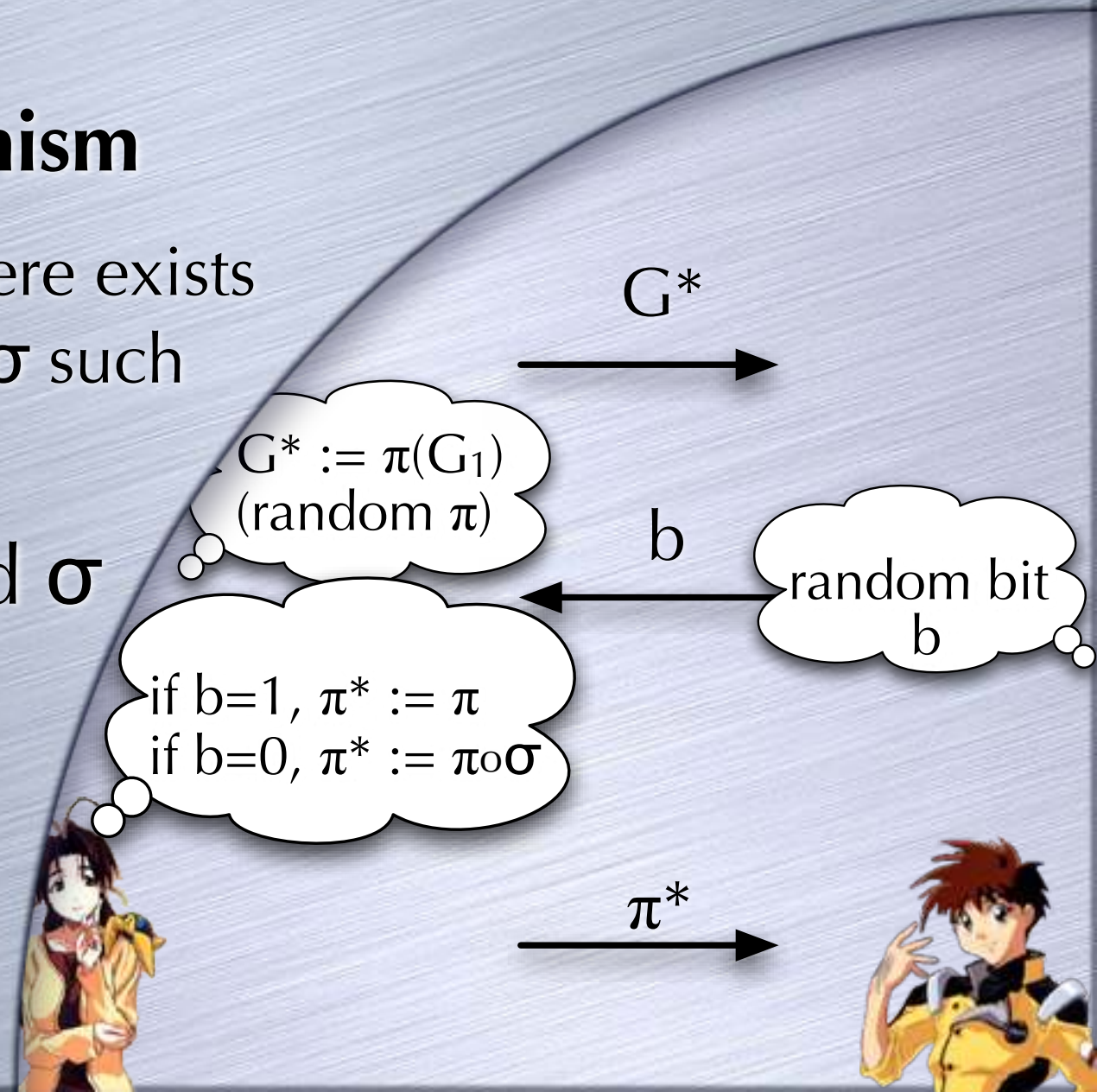
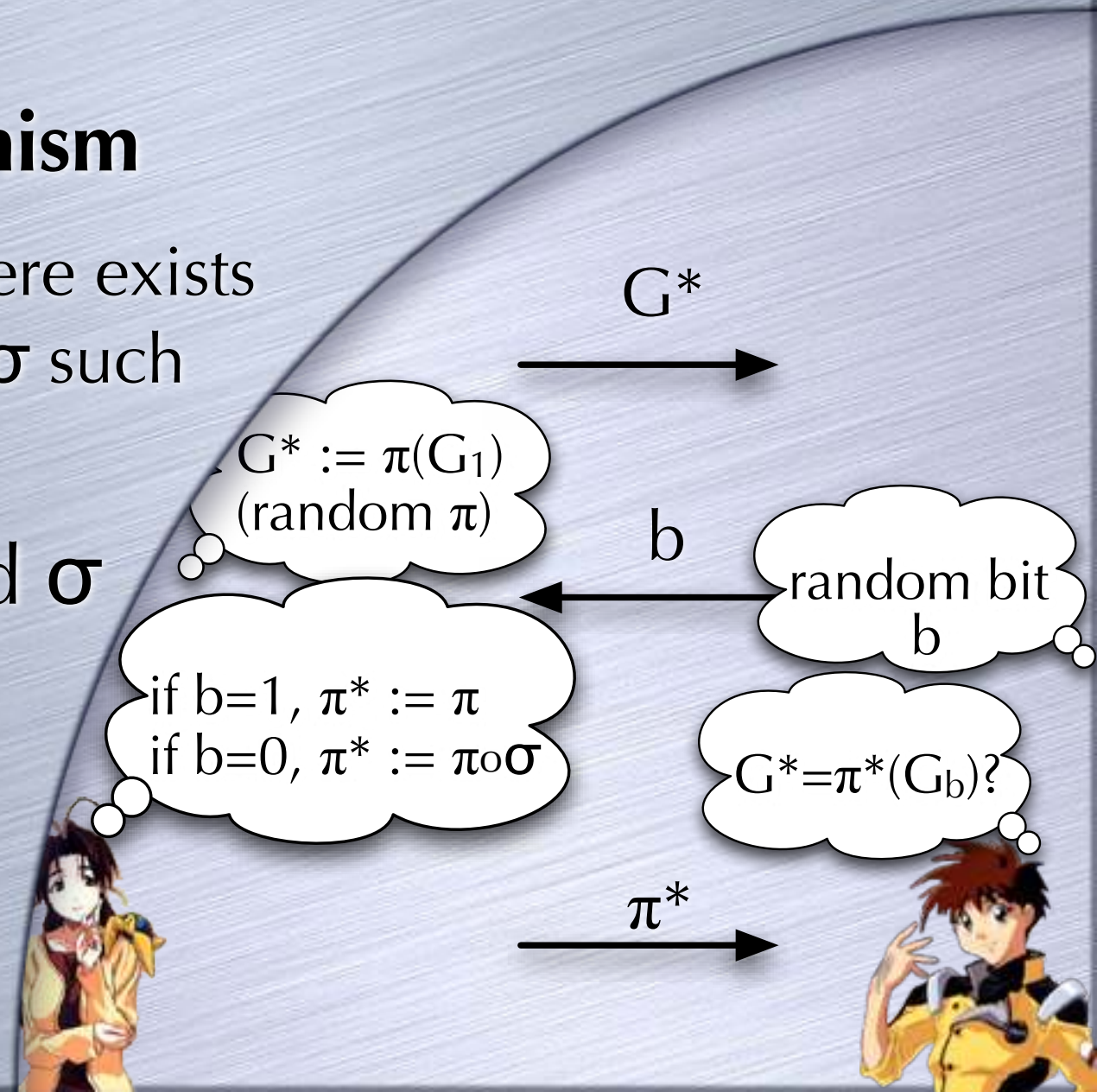


# An Example

- **Graph Isomorphism**
  - $(G_0,G_1)$ in L iff there exists an isomorphism $\sigma$ such that $\sigma(G_0)=G_1$
- IP protocol: send $\sigma$
- ZK protocol?



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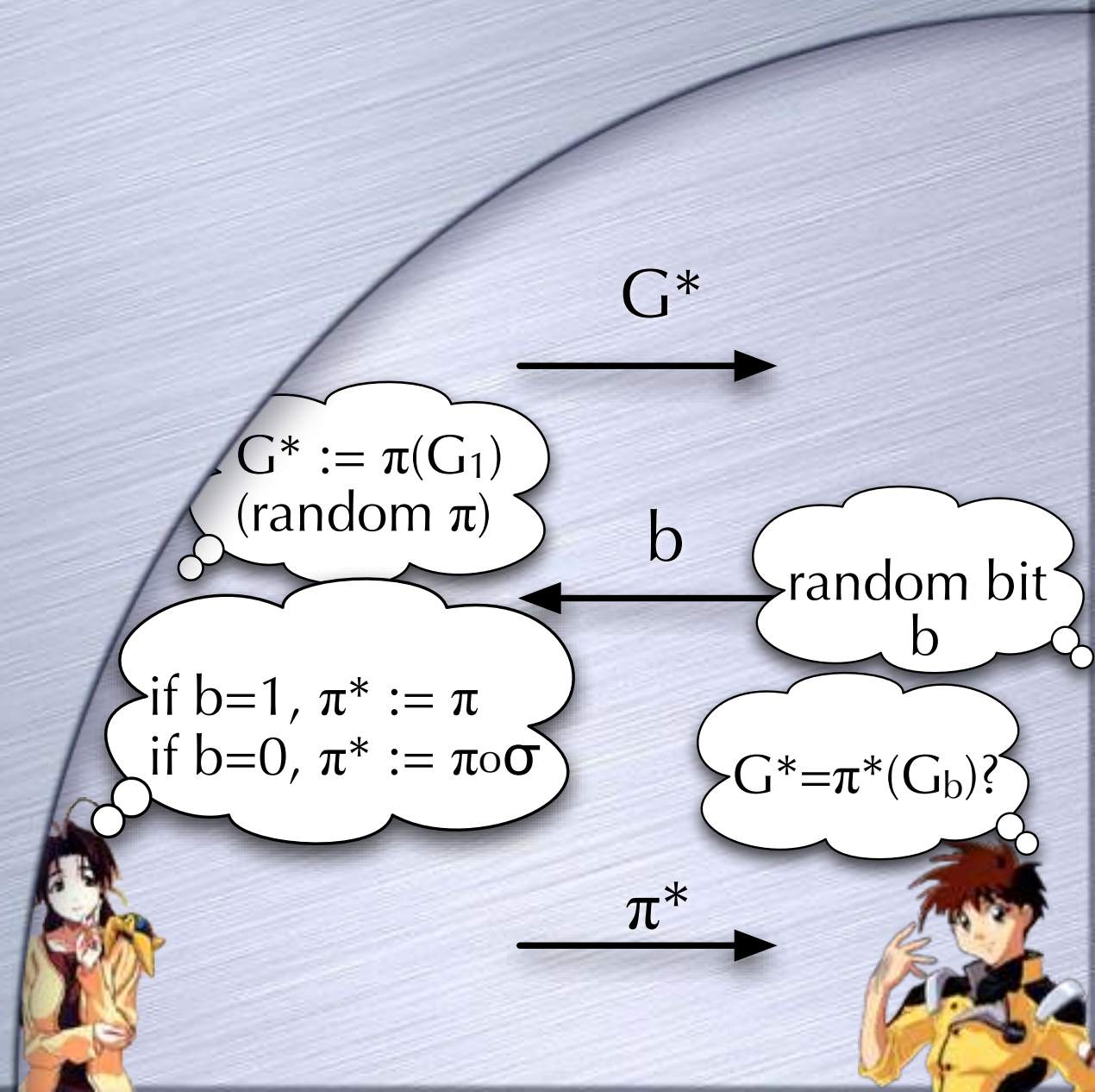

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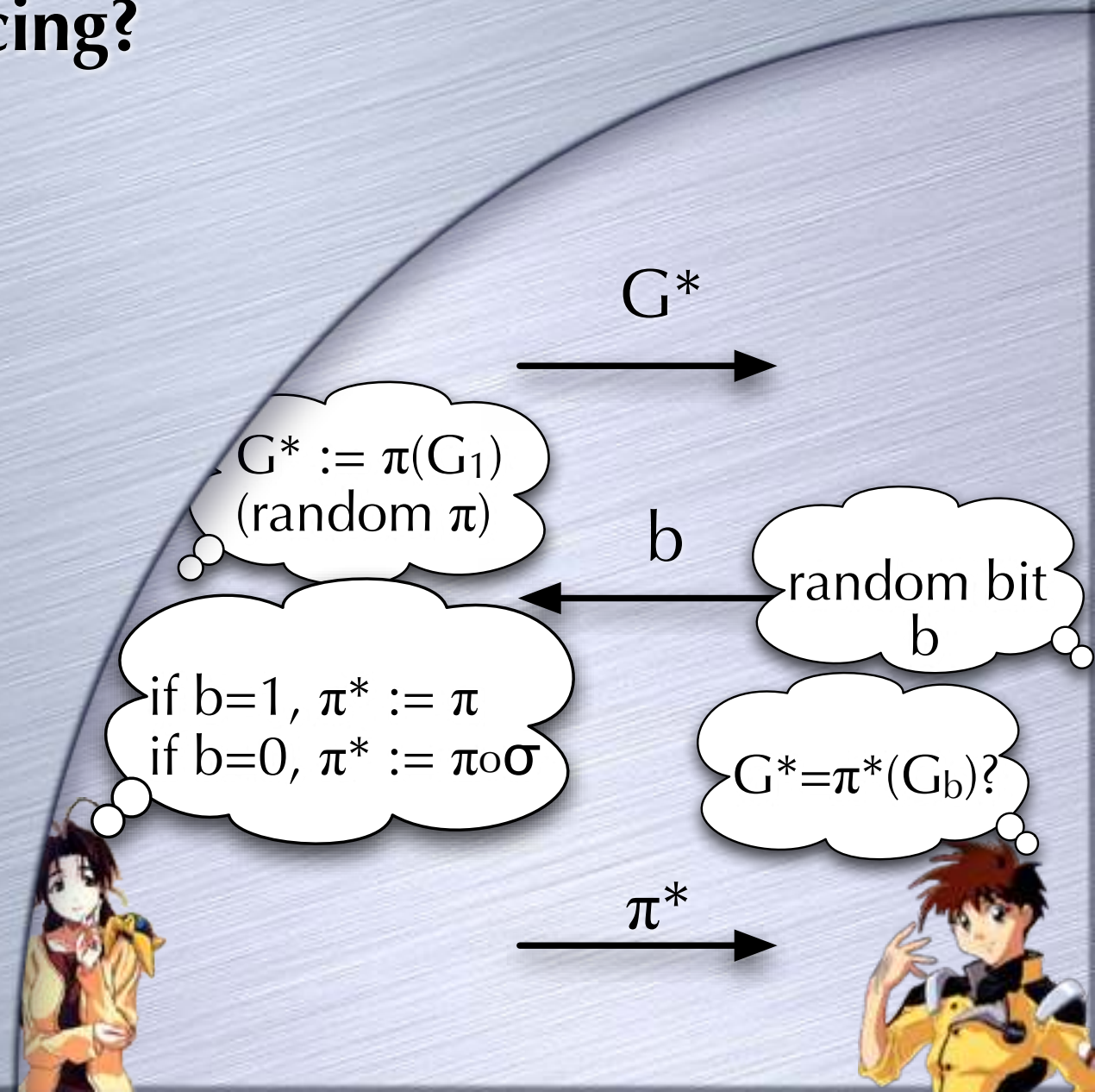

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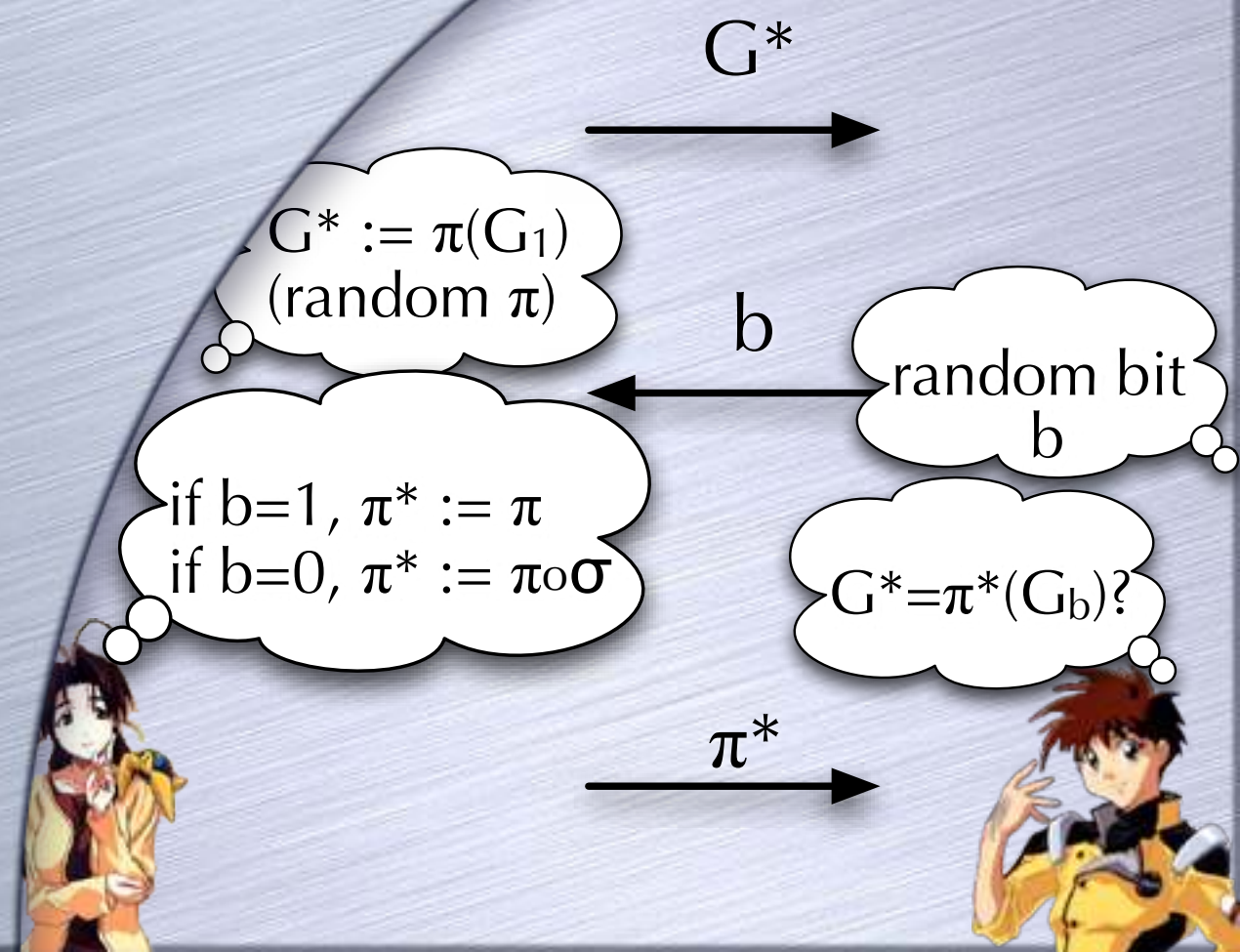

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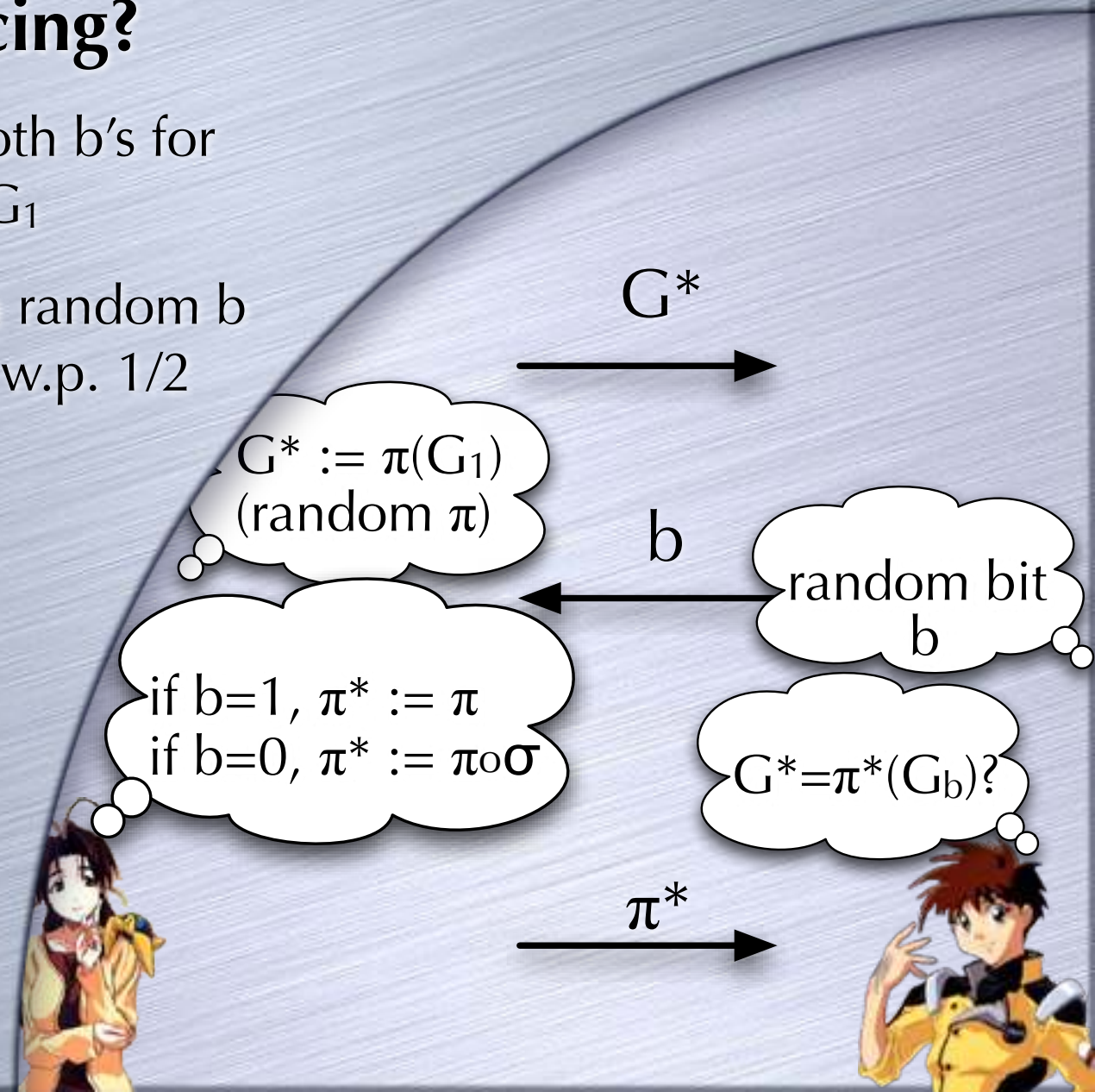

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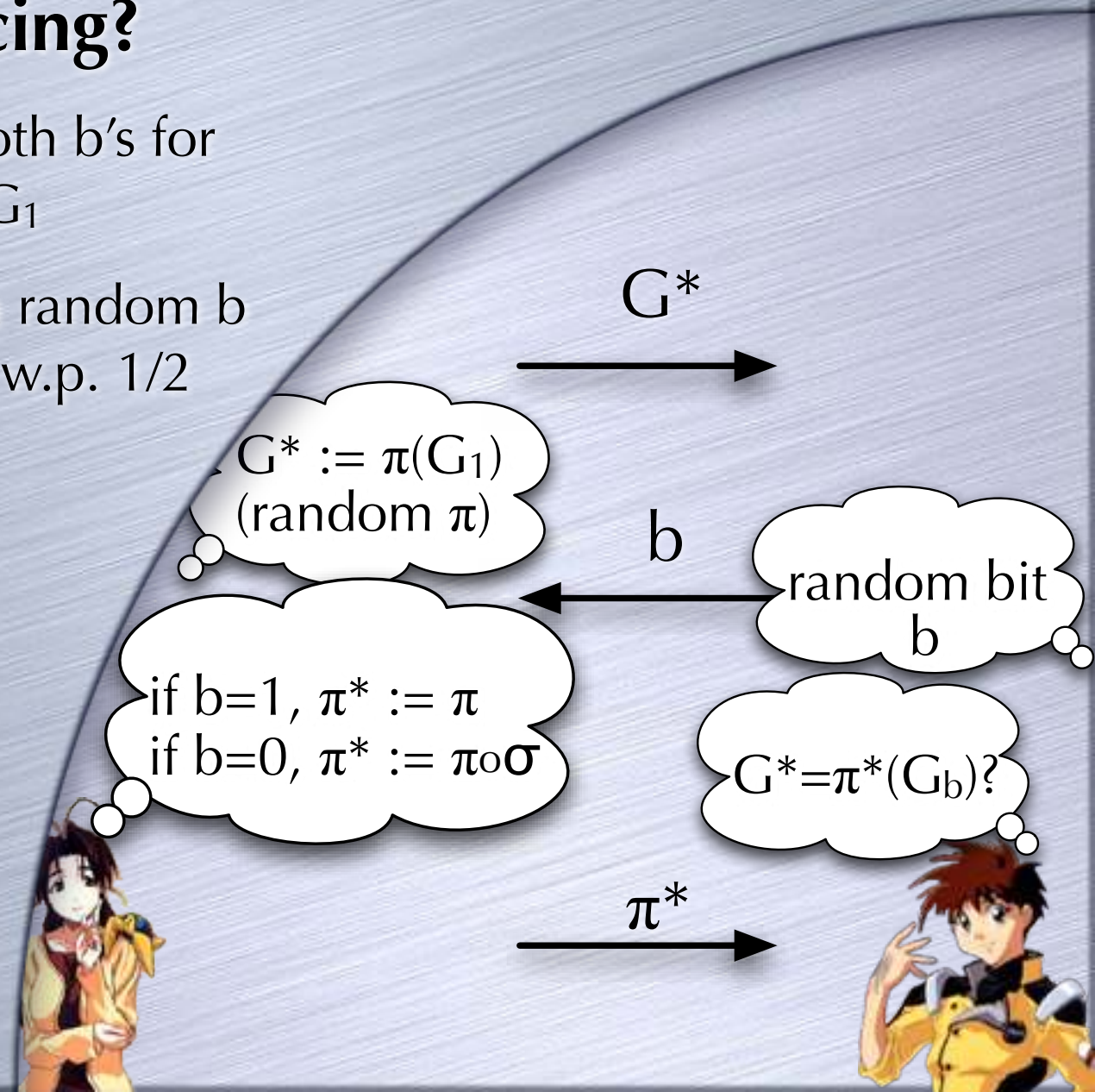

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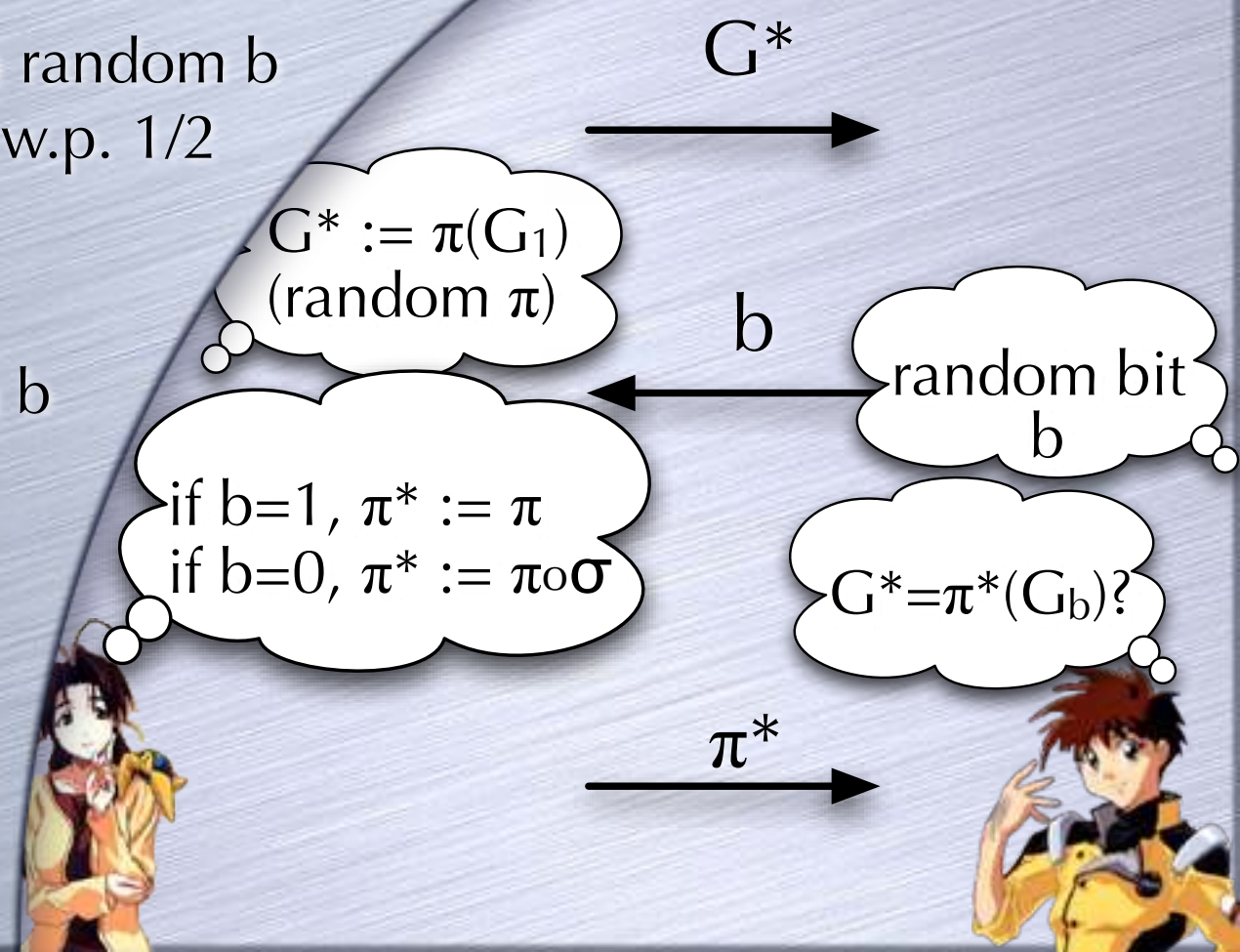

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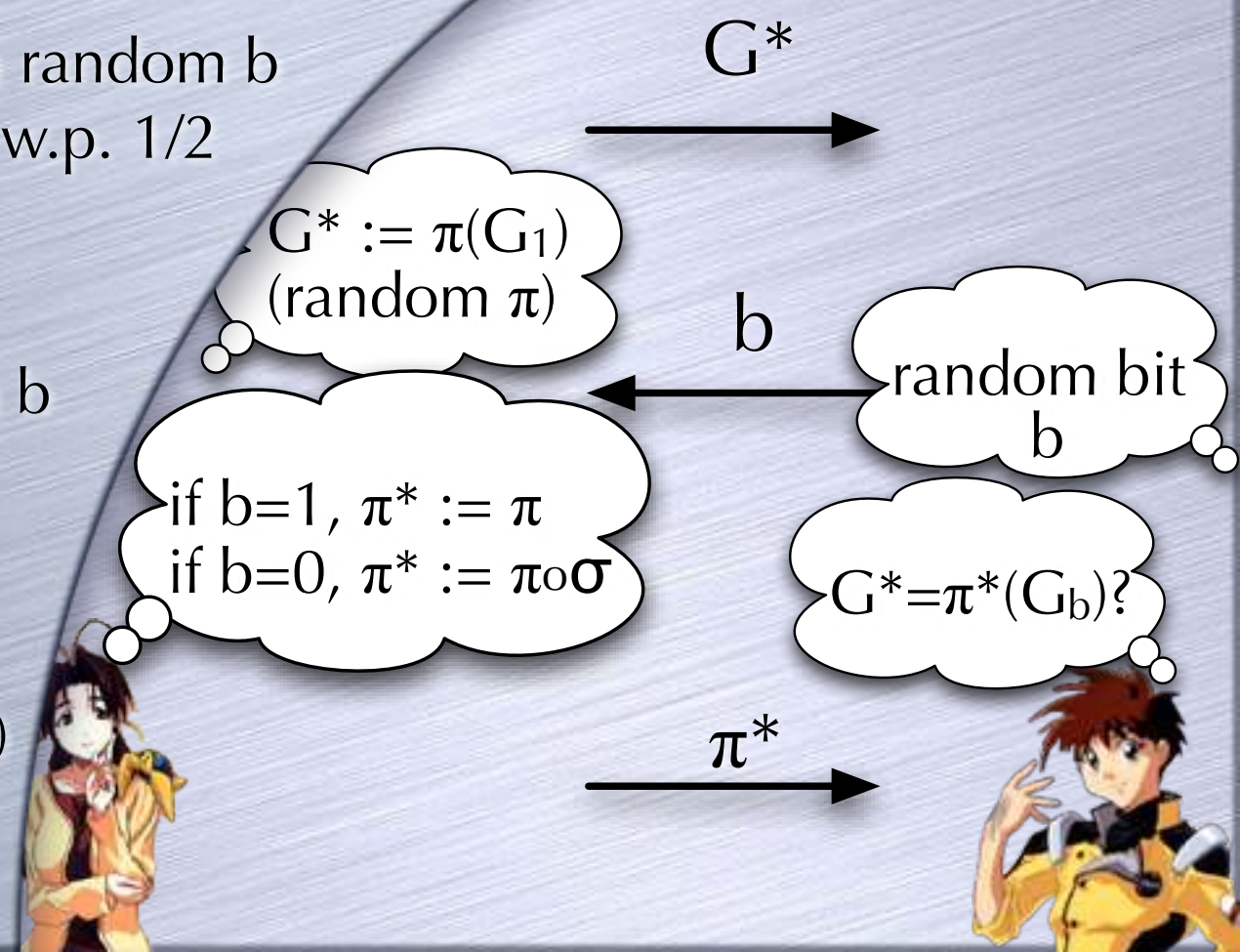

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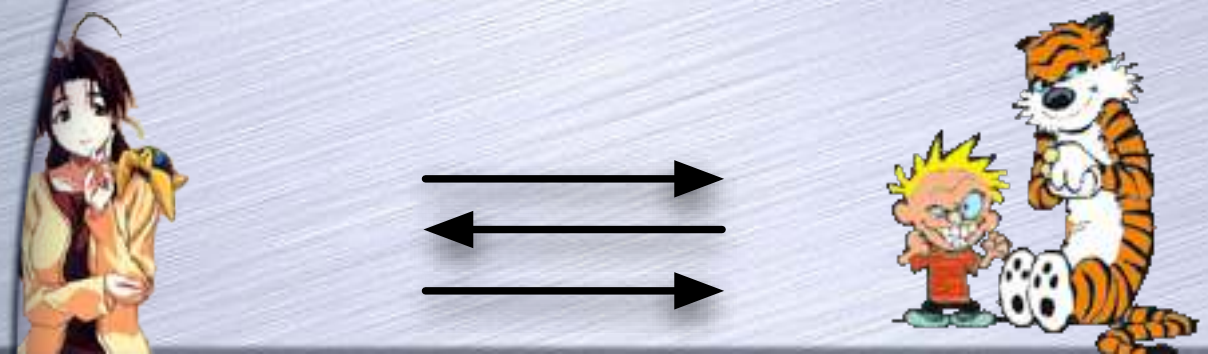

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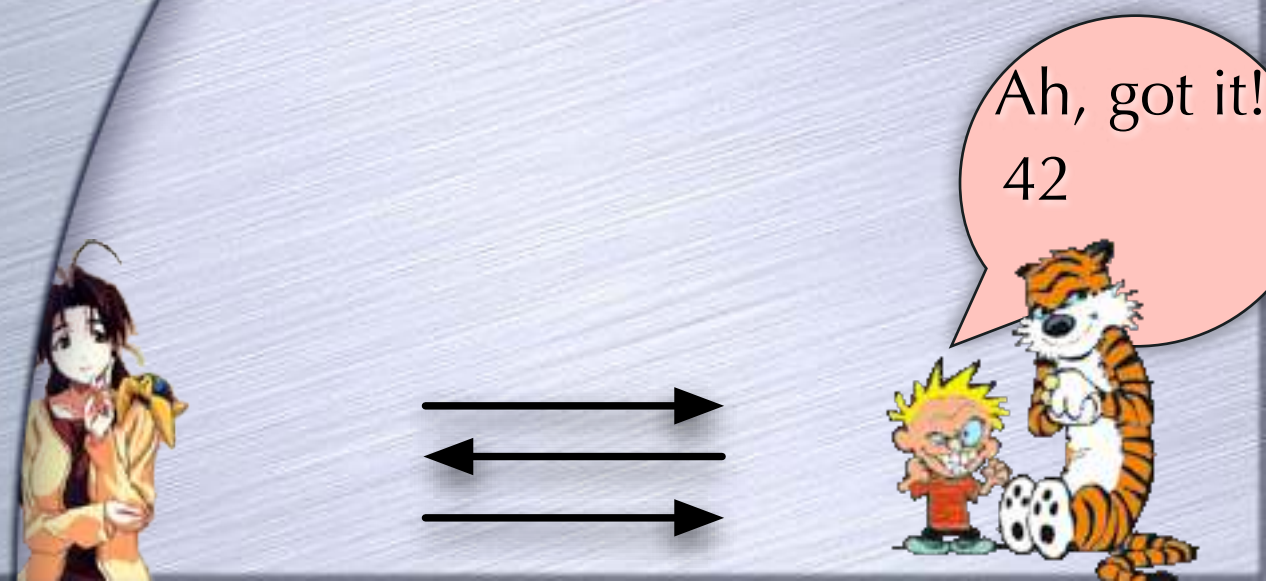

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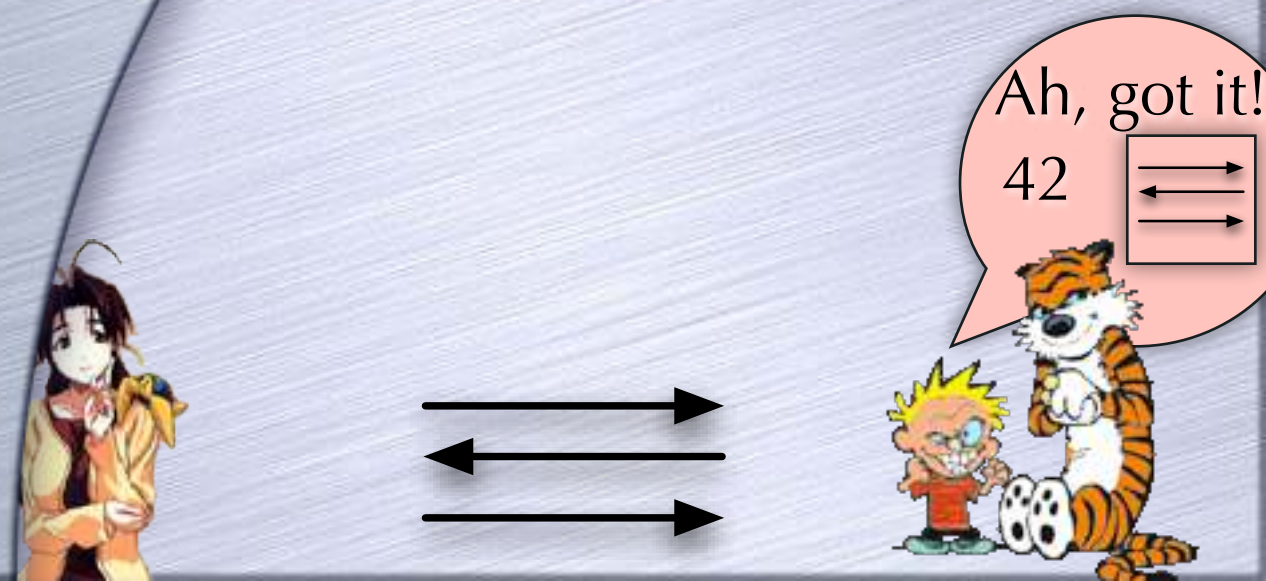

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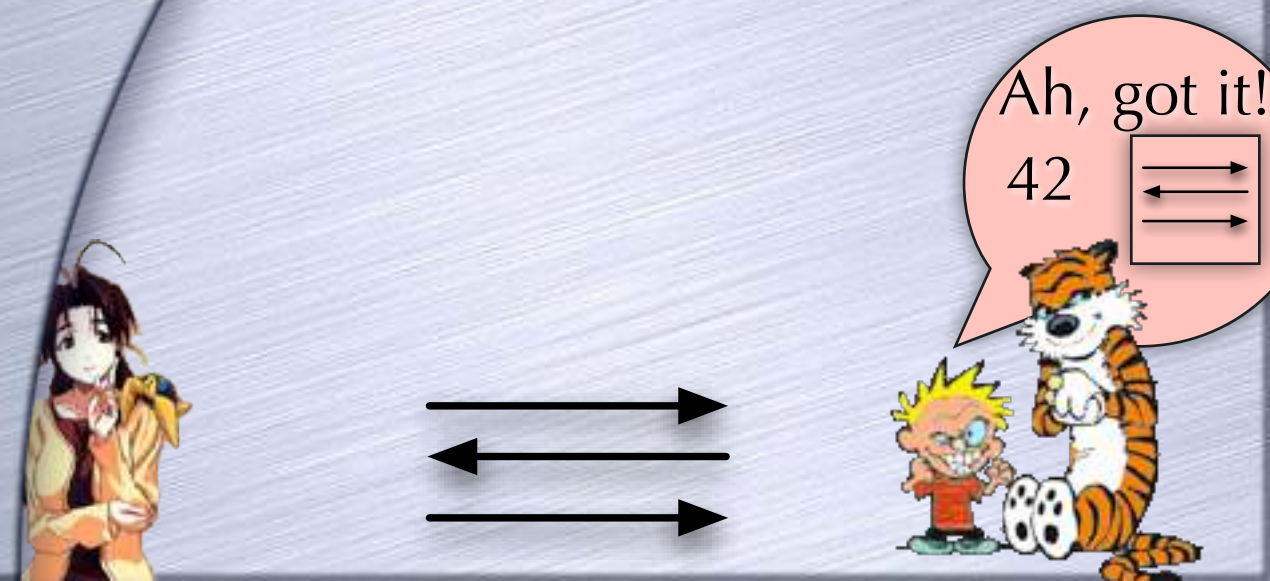

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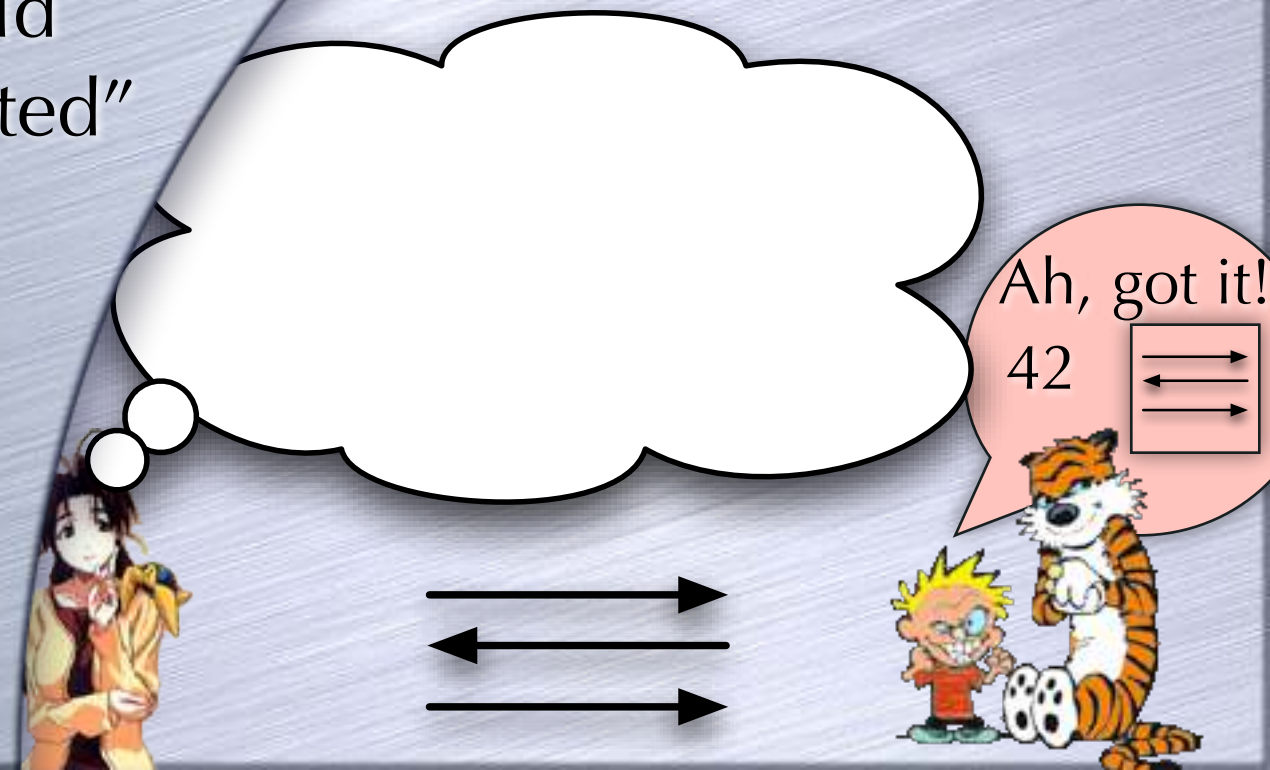

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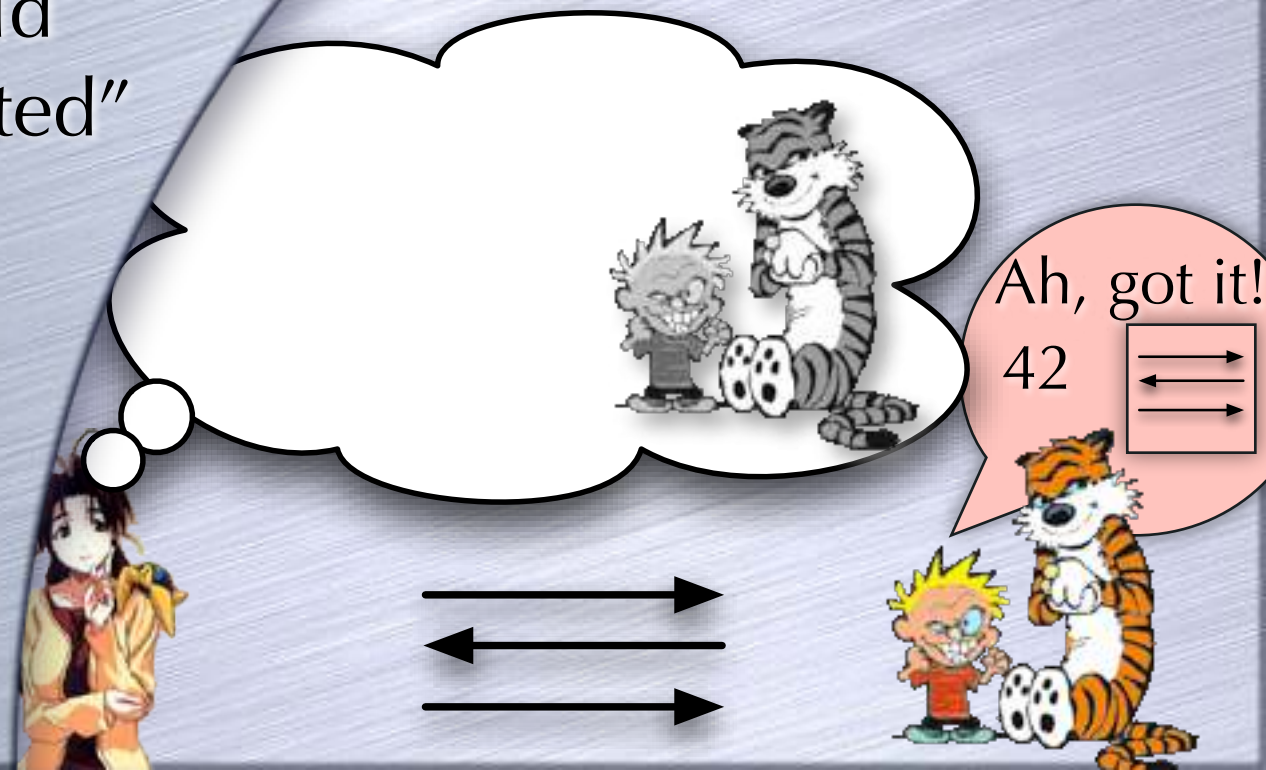

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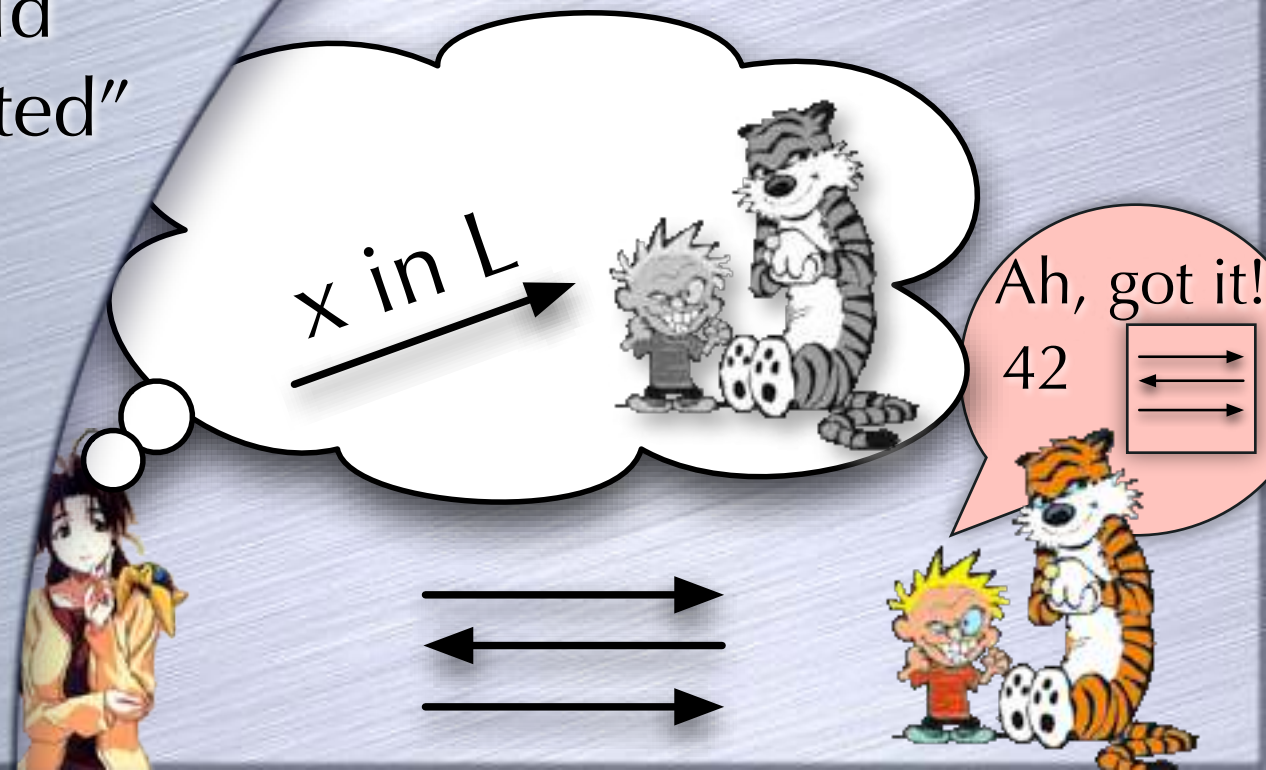

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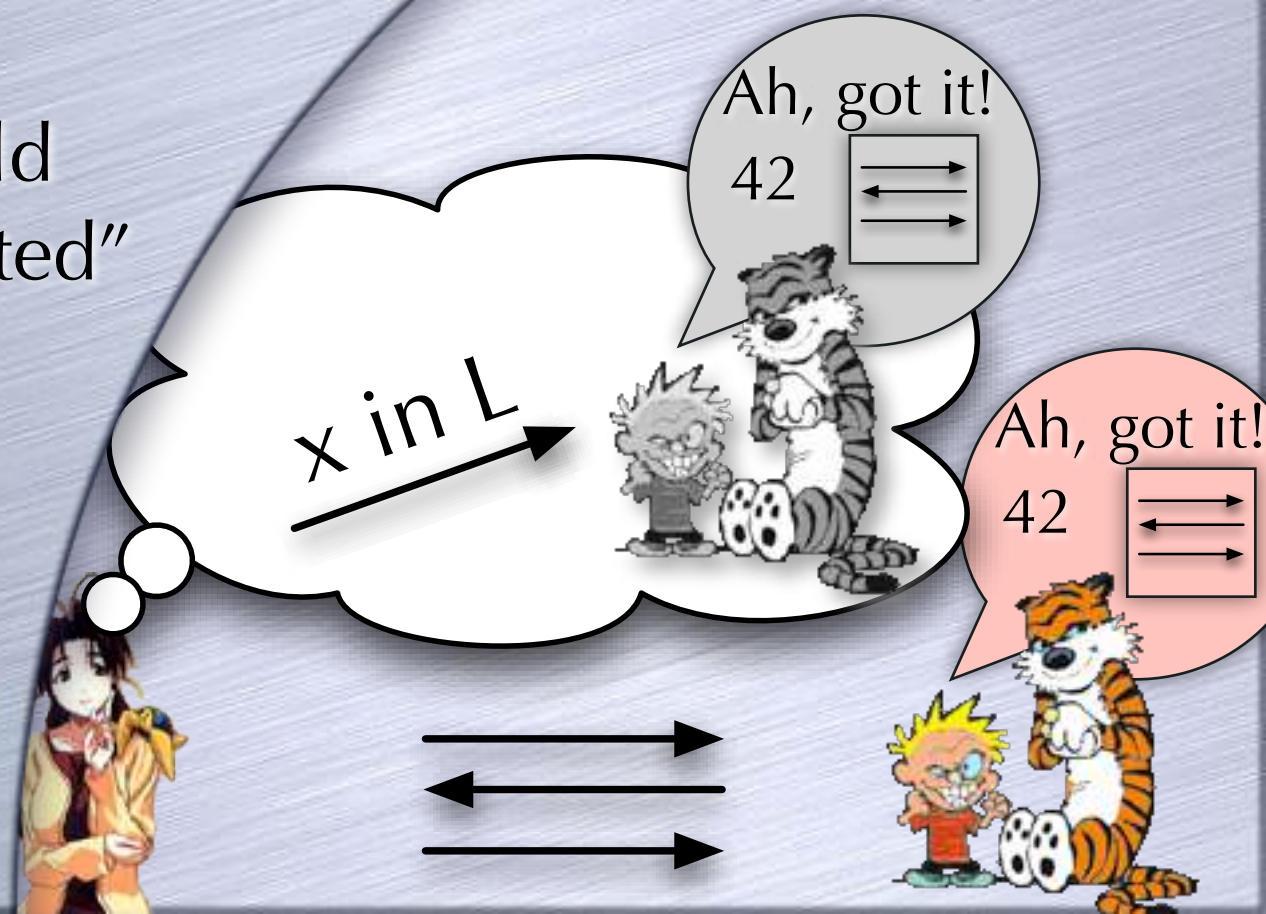

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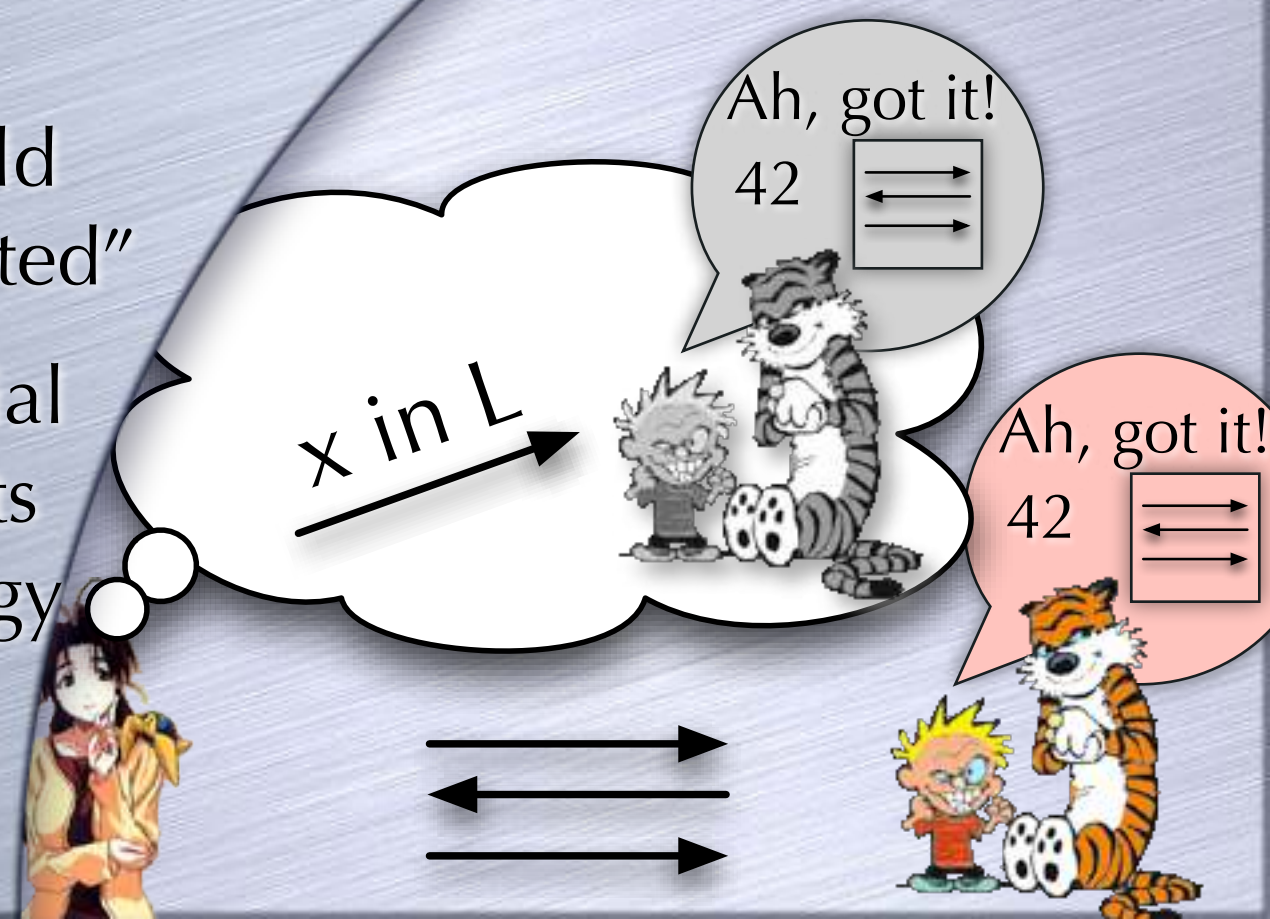
- ZK Property:
  - Verifier's view could have been "simulated"
  - For every adversarial strategy, there exists a simulation strategy



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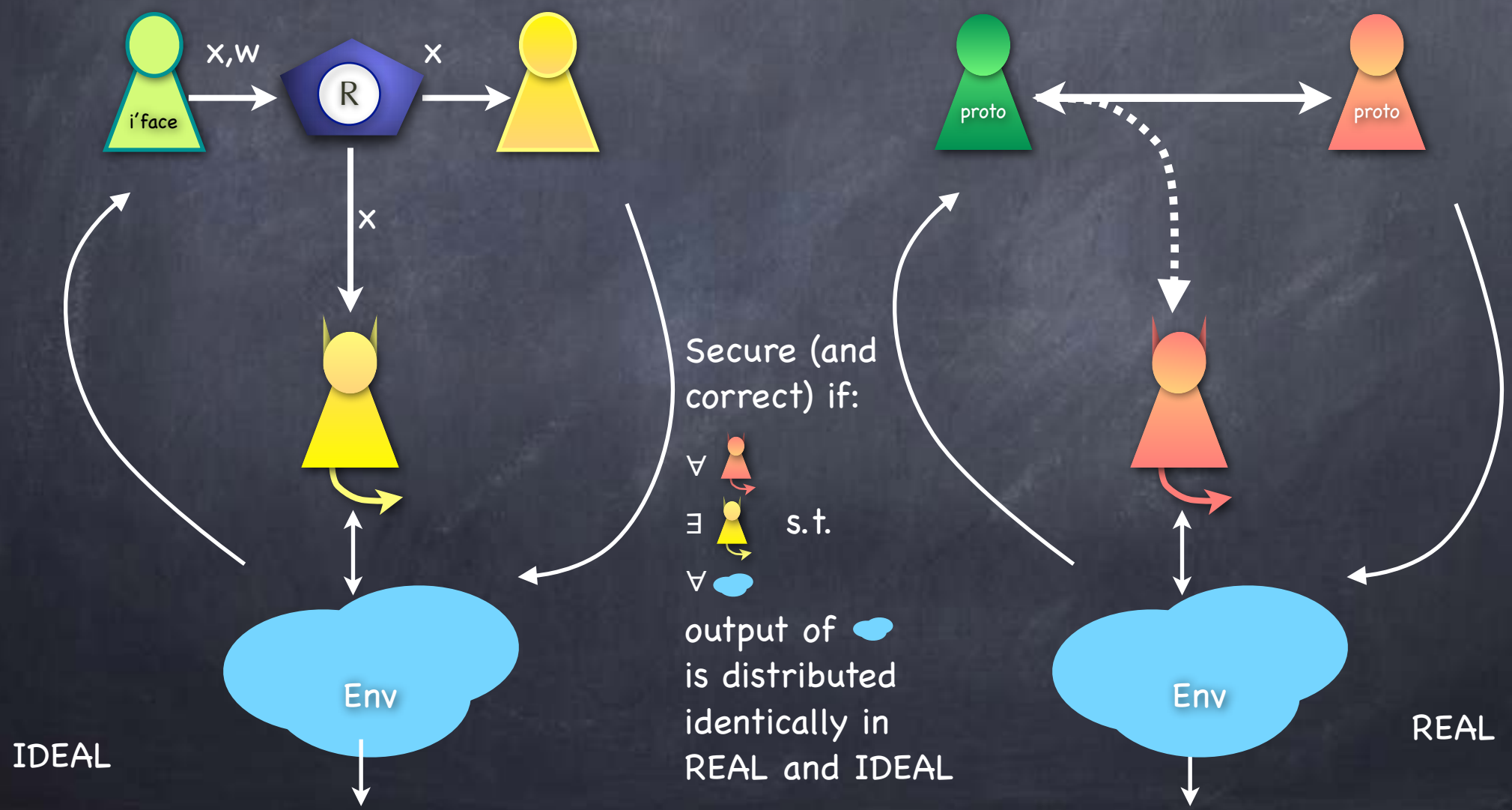

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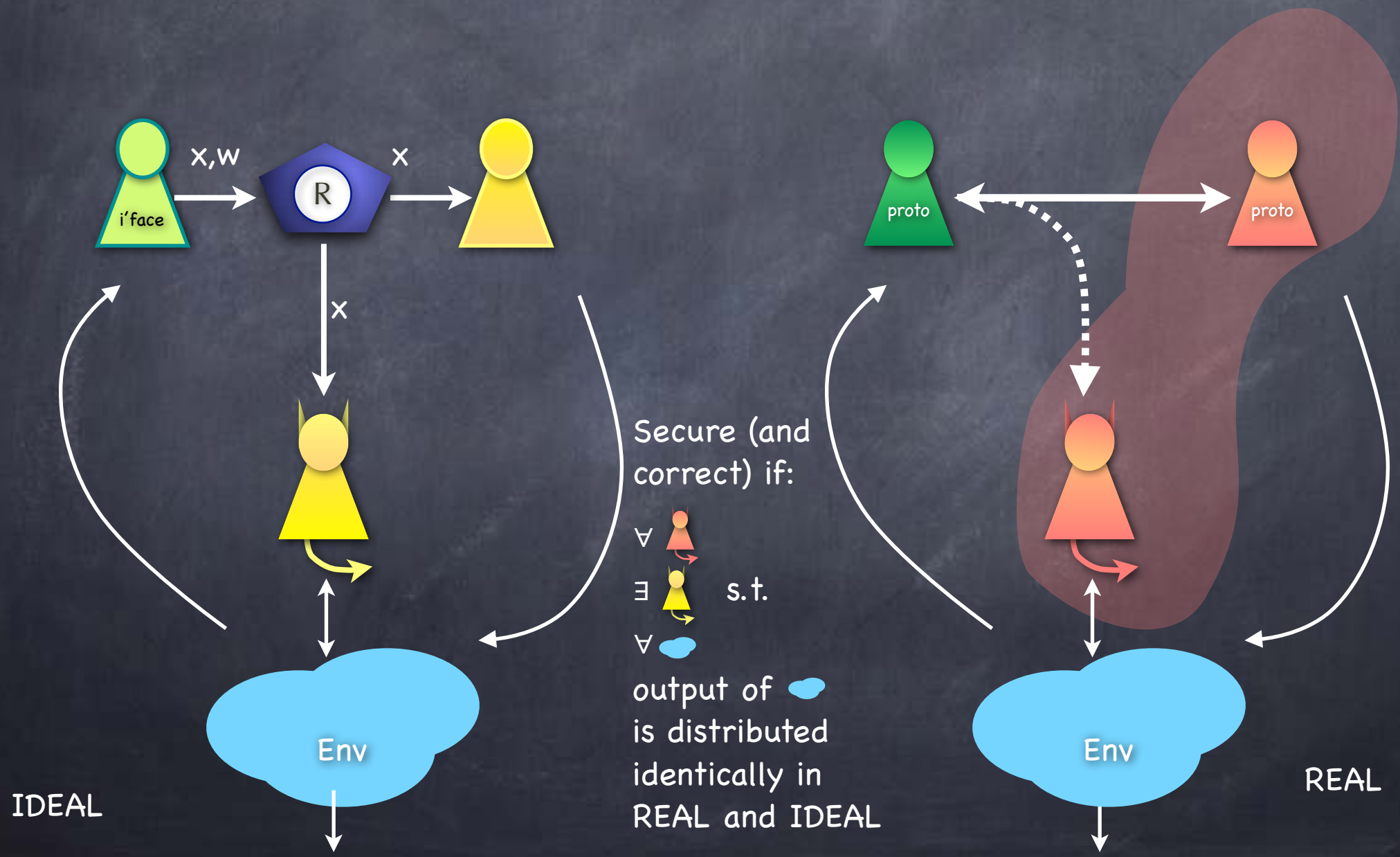

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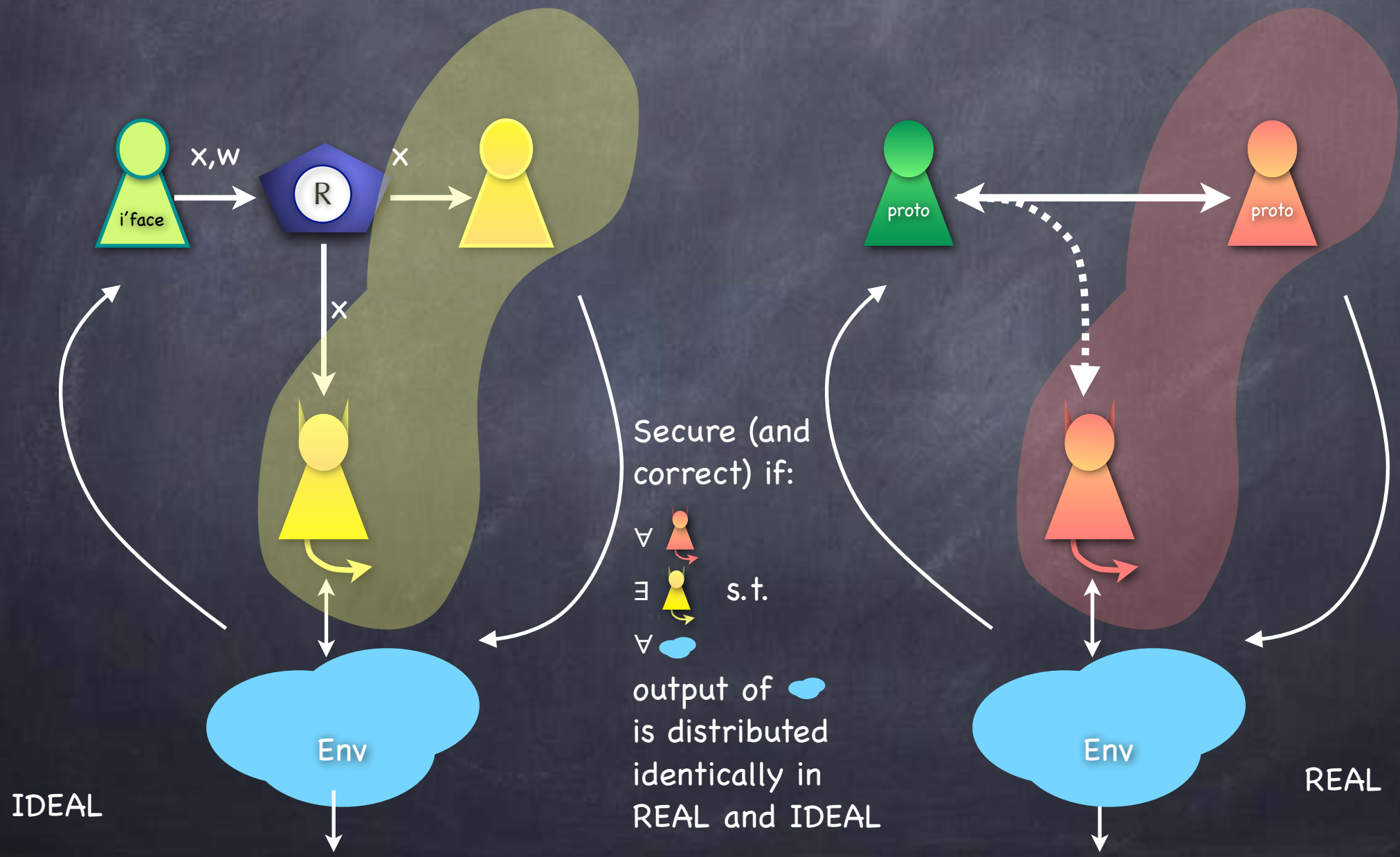
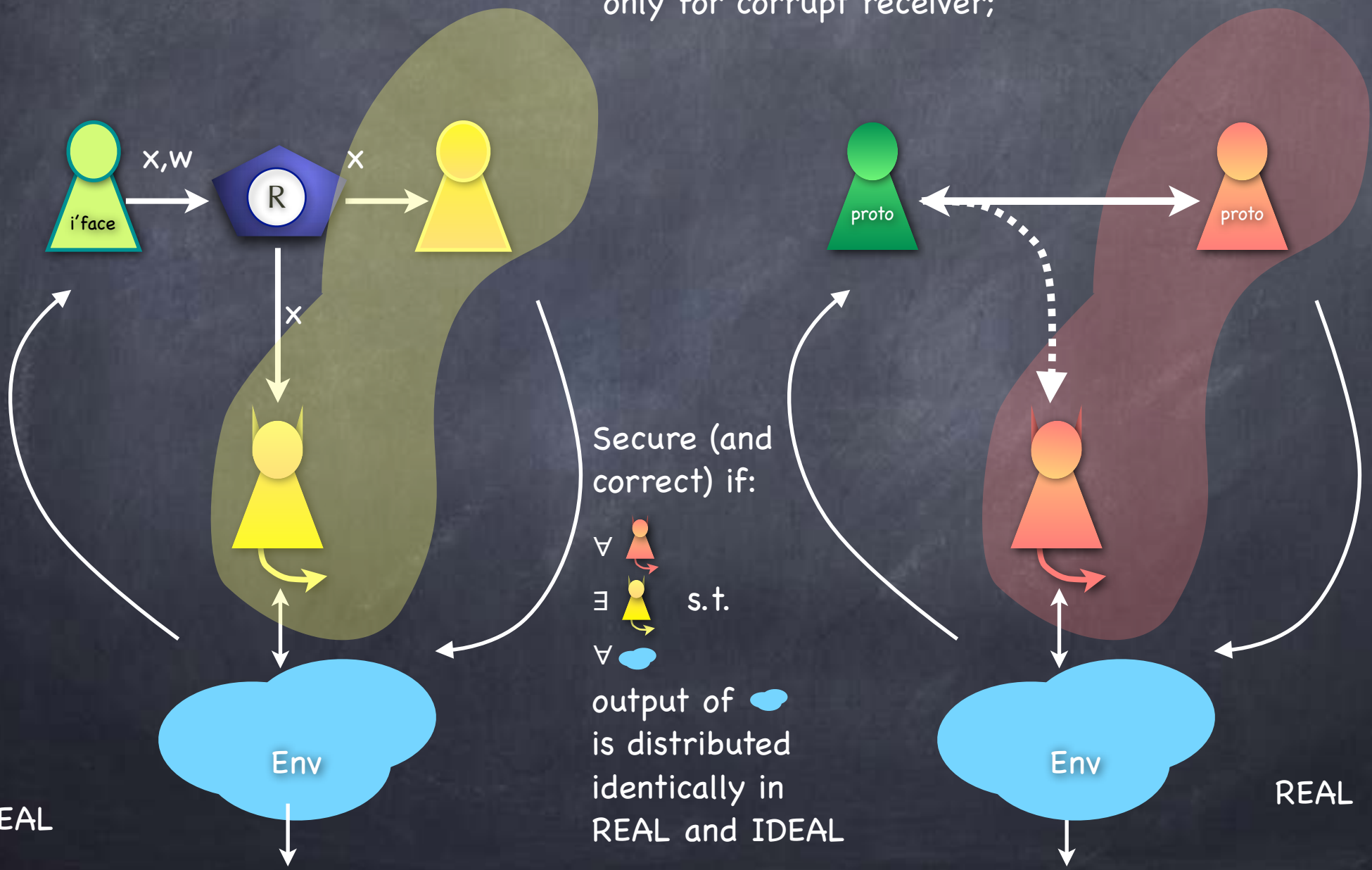


# ZK Property (in other pics)

Classical definition uses simulation only for corrupt receiver;

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∀ ∃ s.t. ∀

output of is distributed identically in REAL and IDEAL

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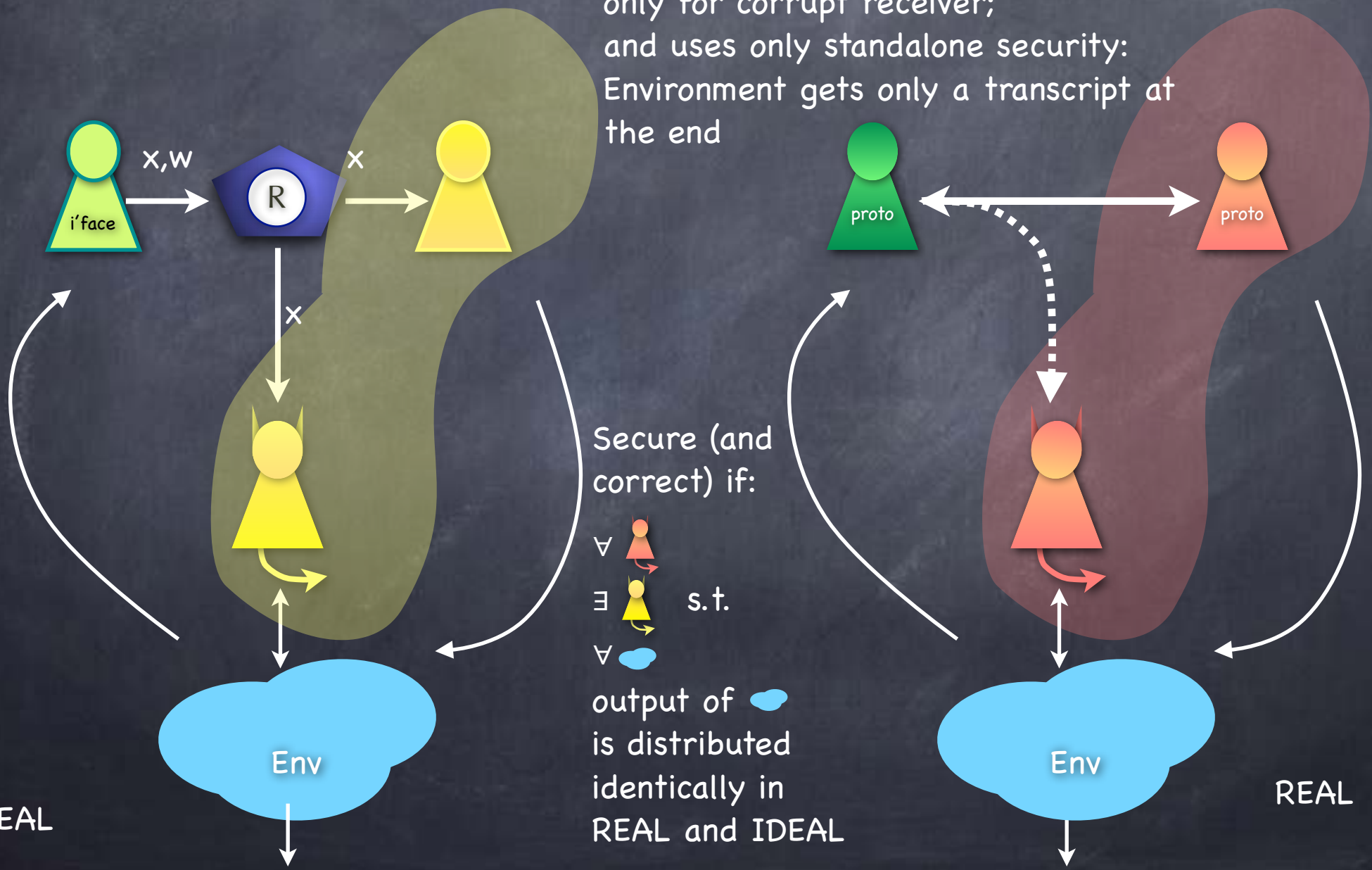
Classical definition uses simulation only for corrupt receiver;
and uses only standalone security:
Environment gets only a transcript at the end

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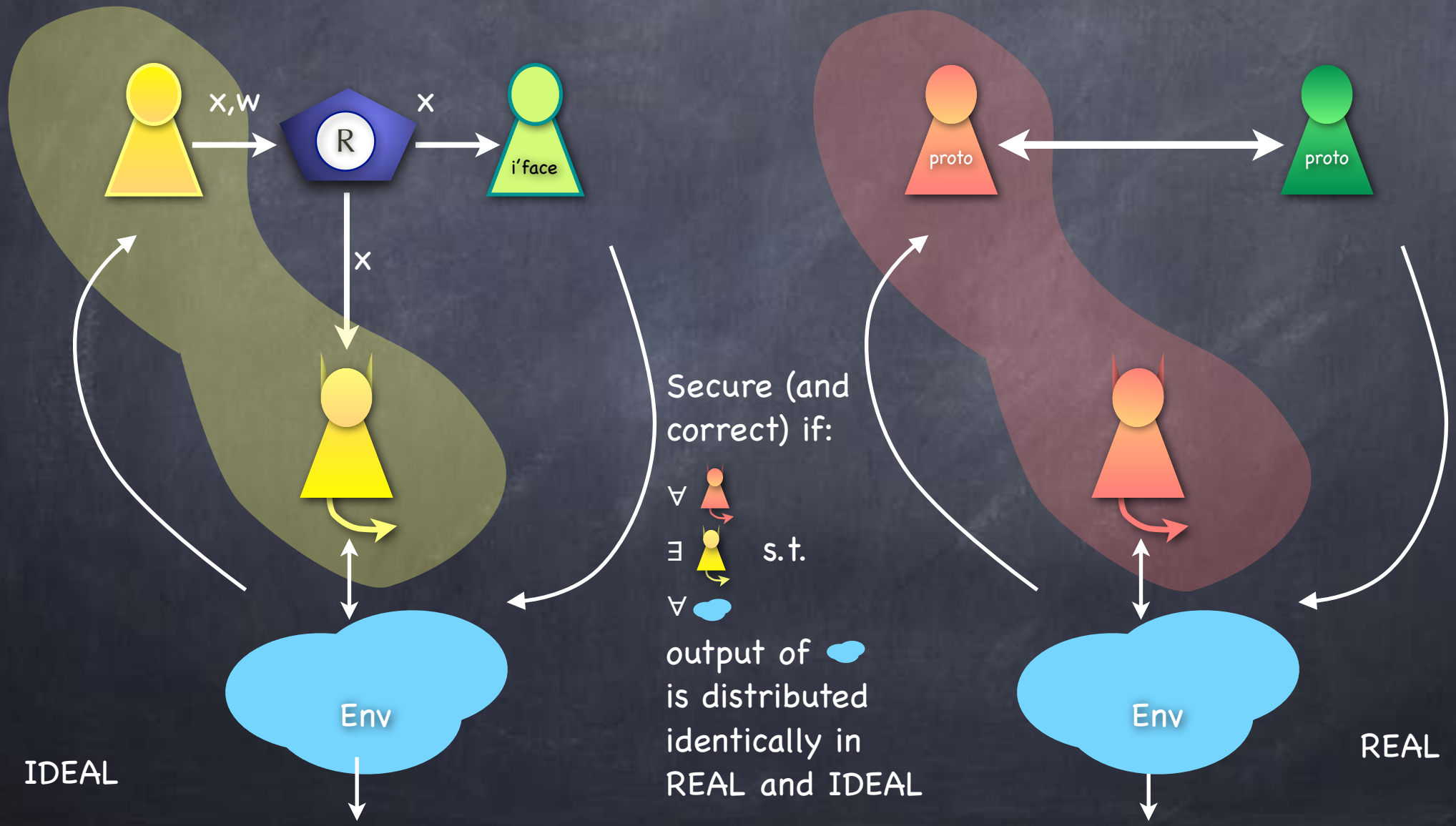


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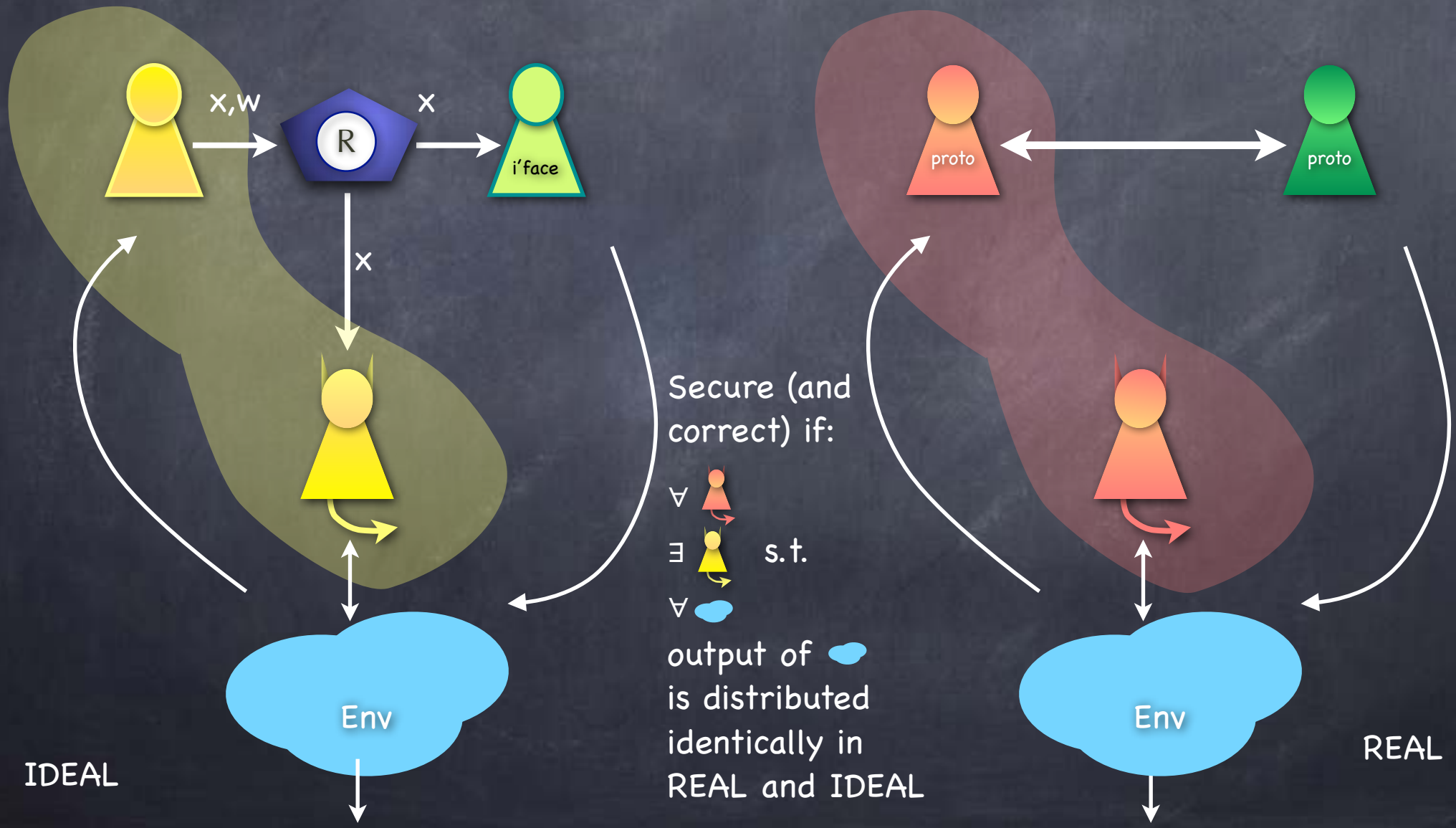


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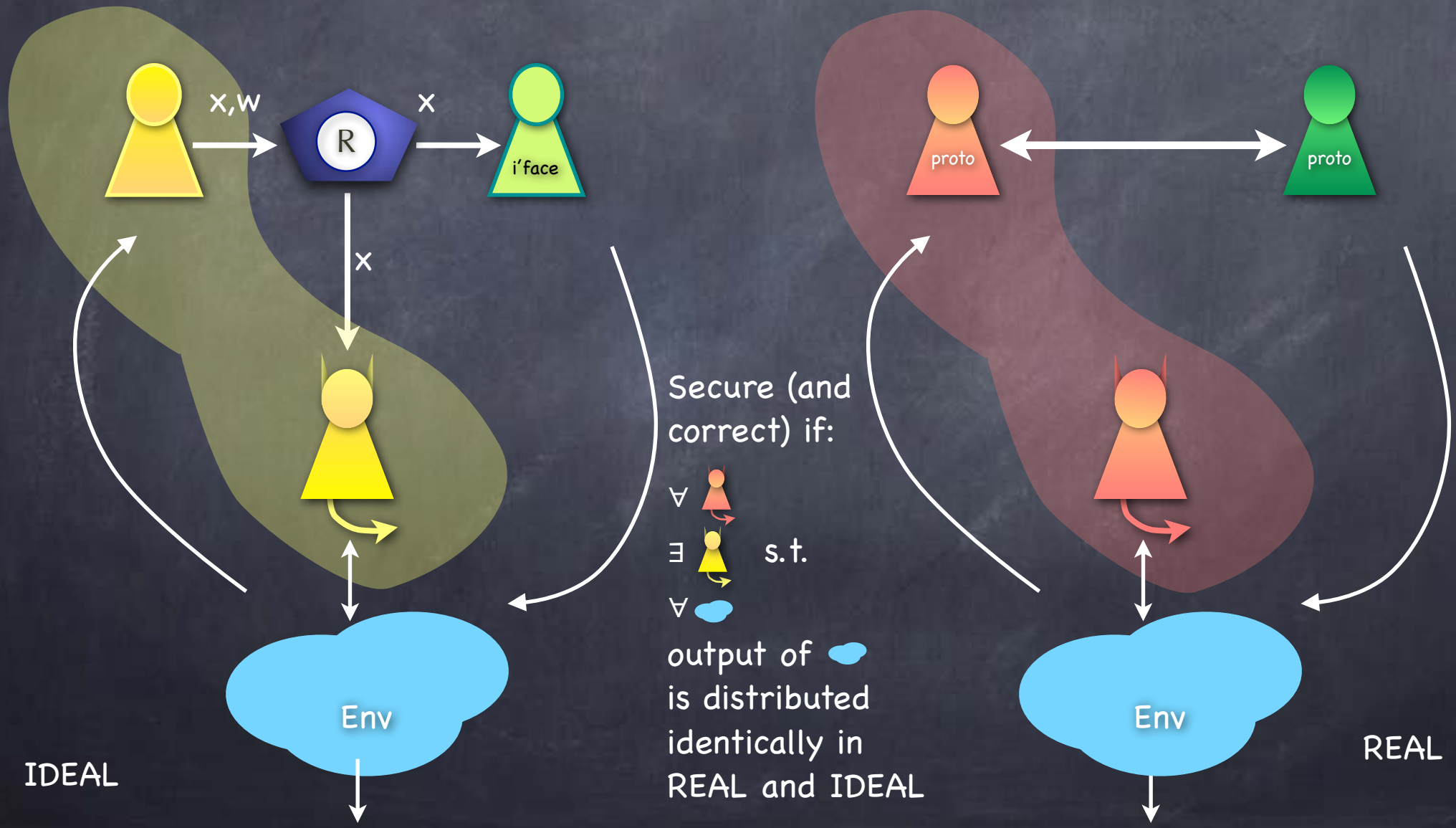

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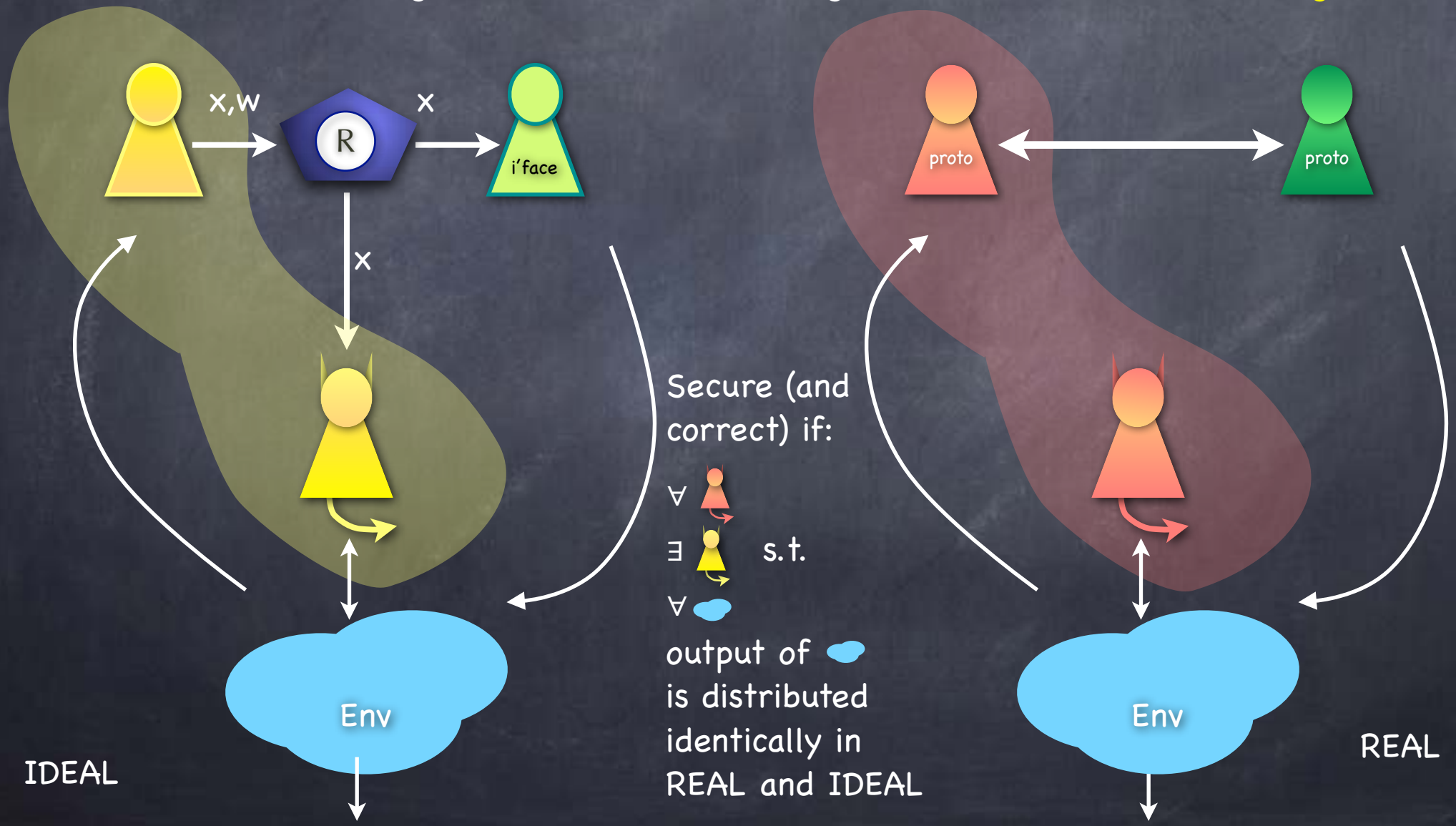
- Adding this (in standalone setting) makes it a Proof of Knowledge



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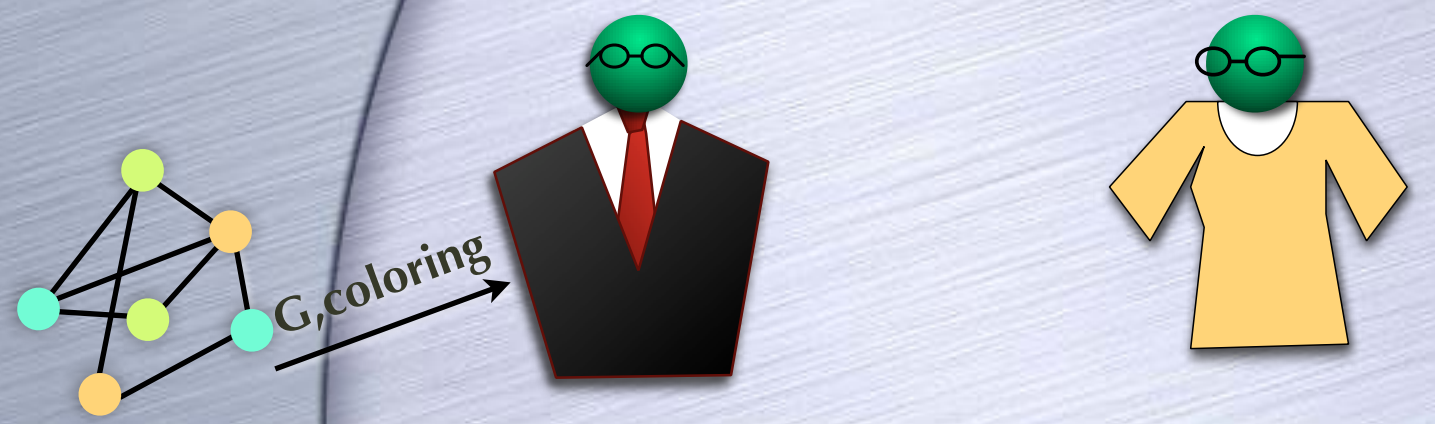

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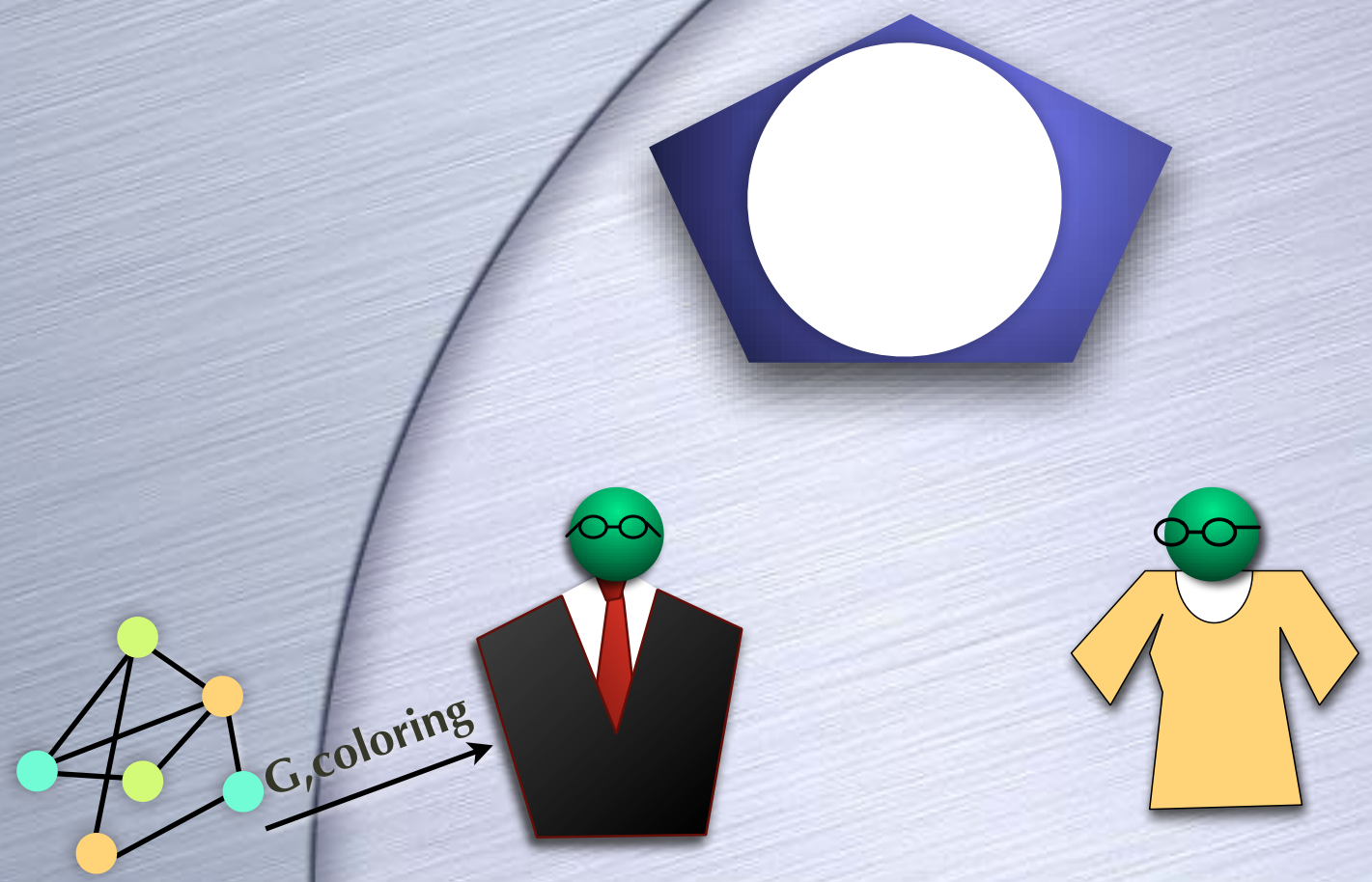

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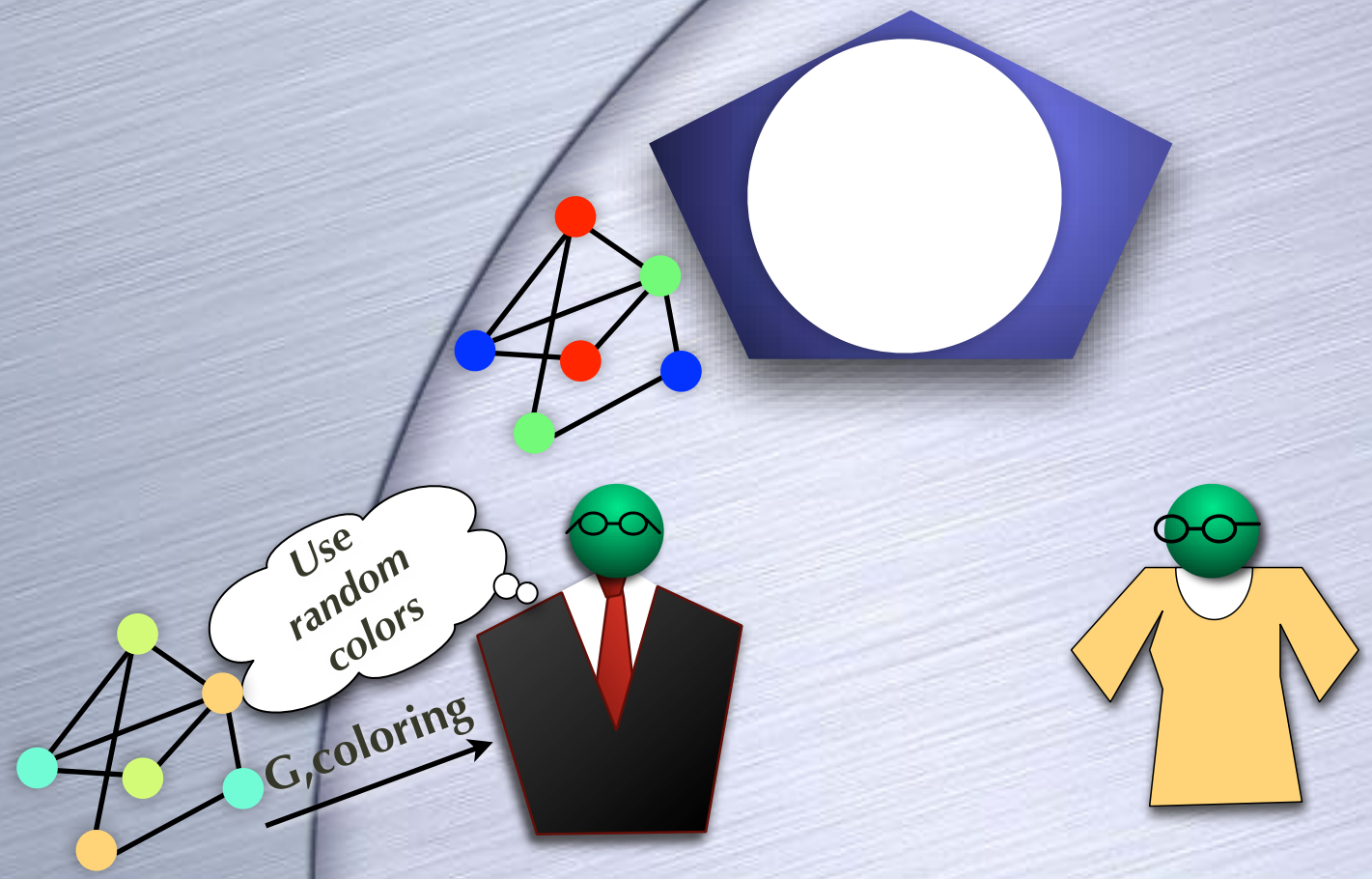

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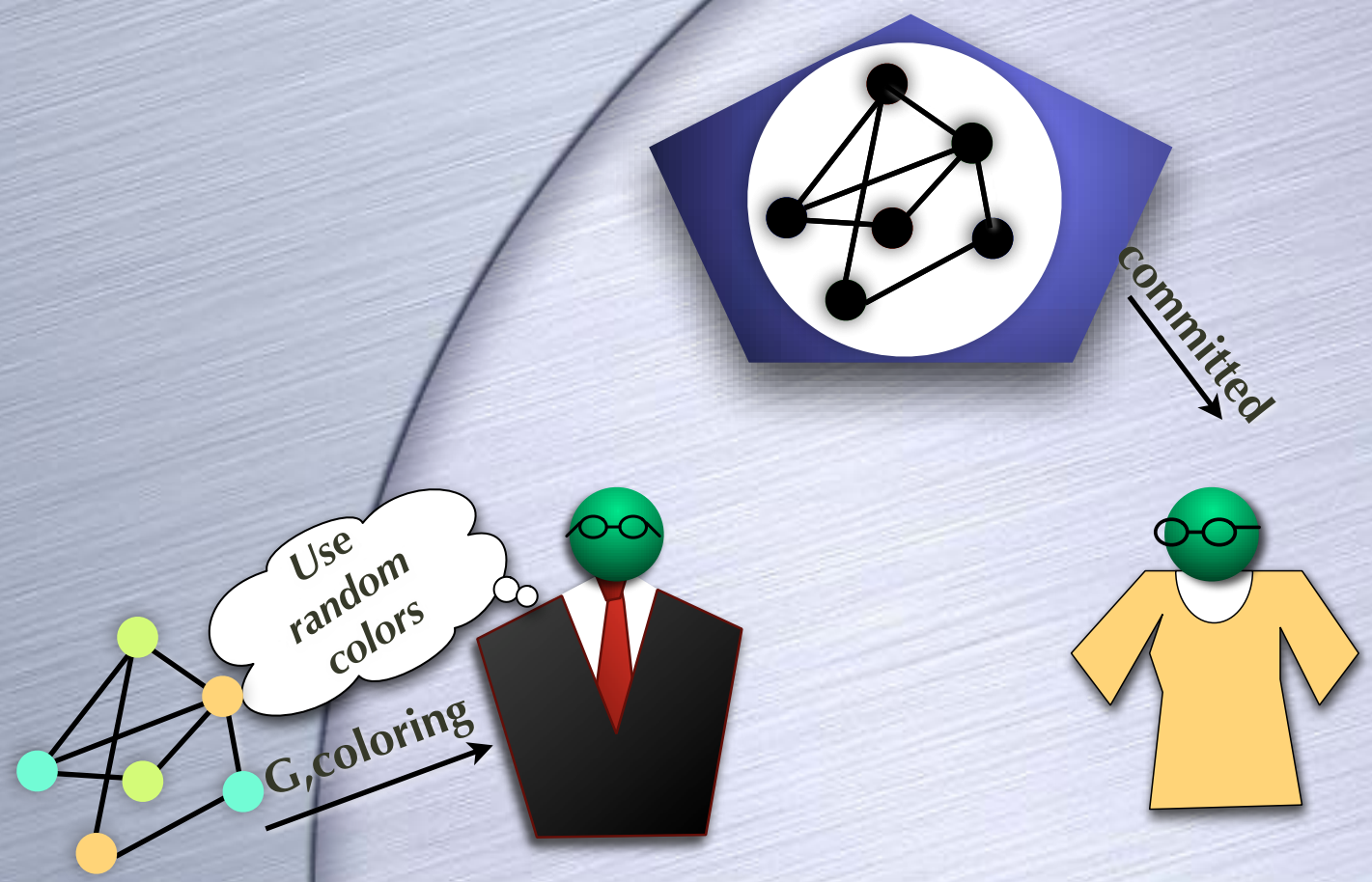

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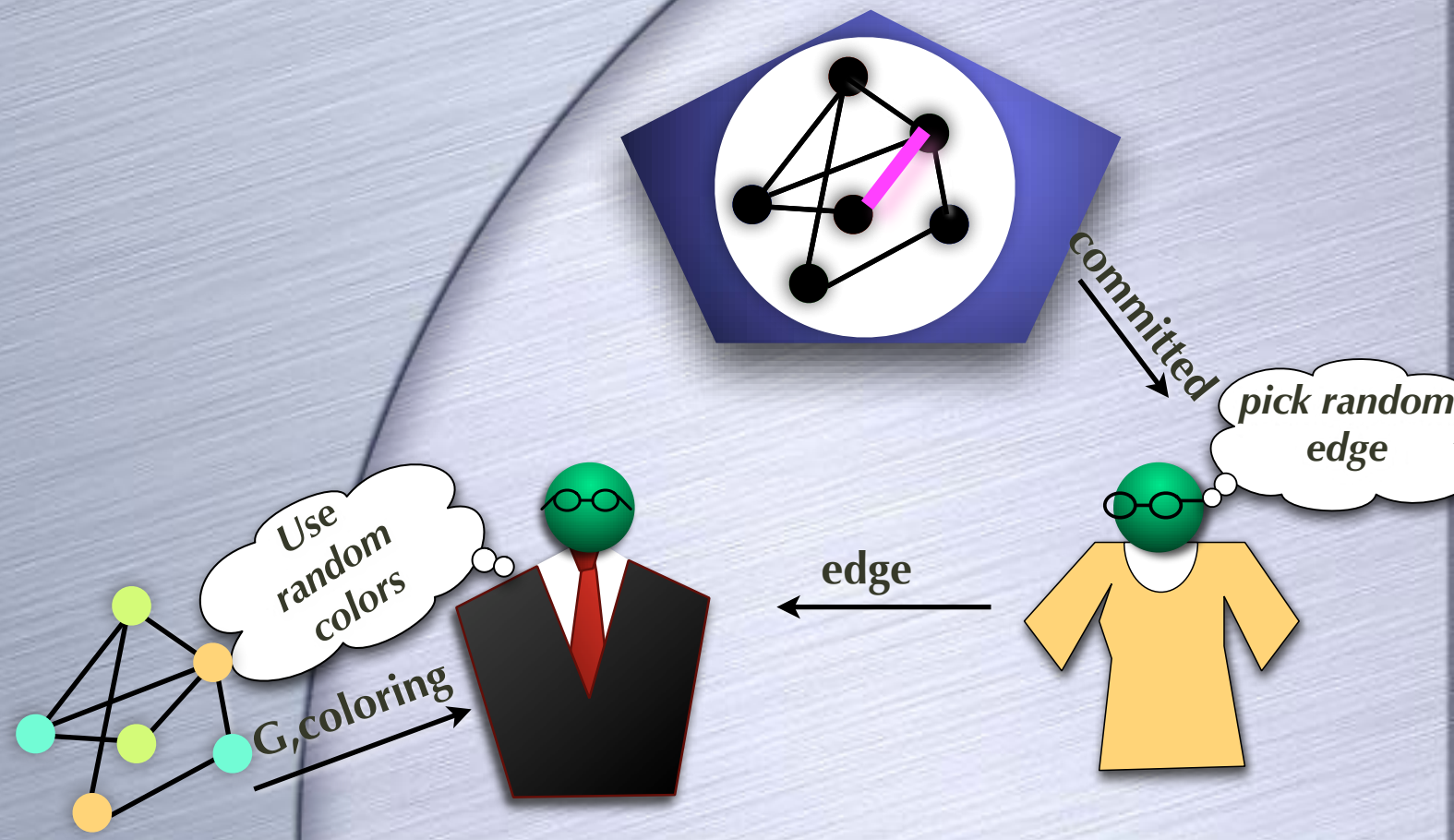

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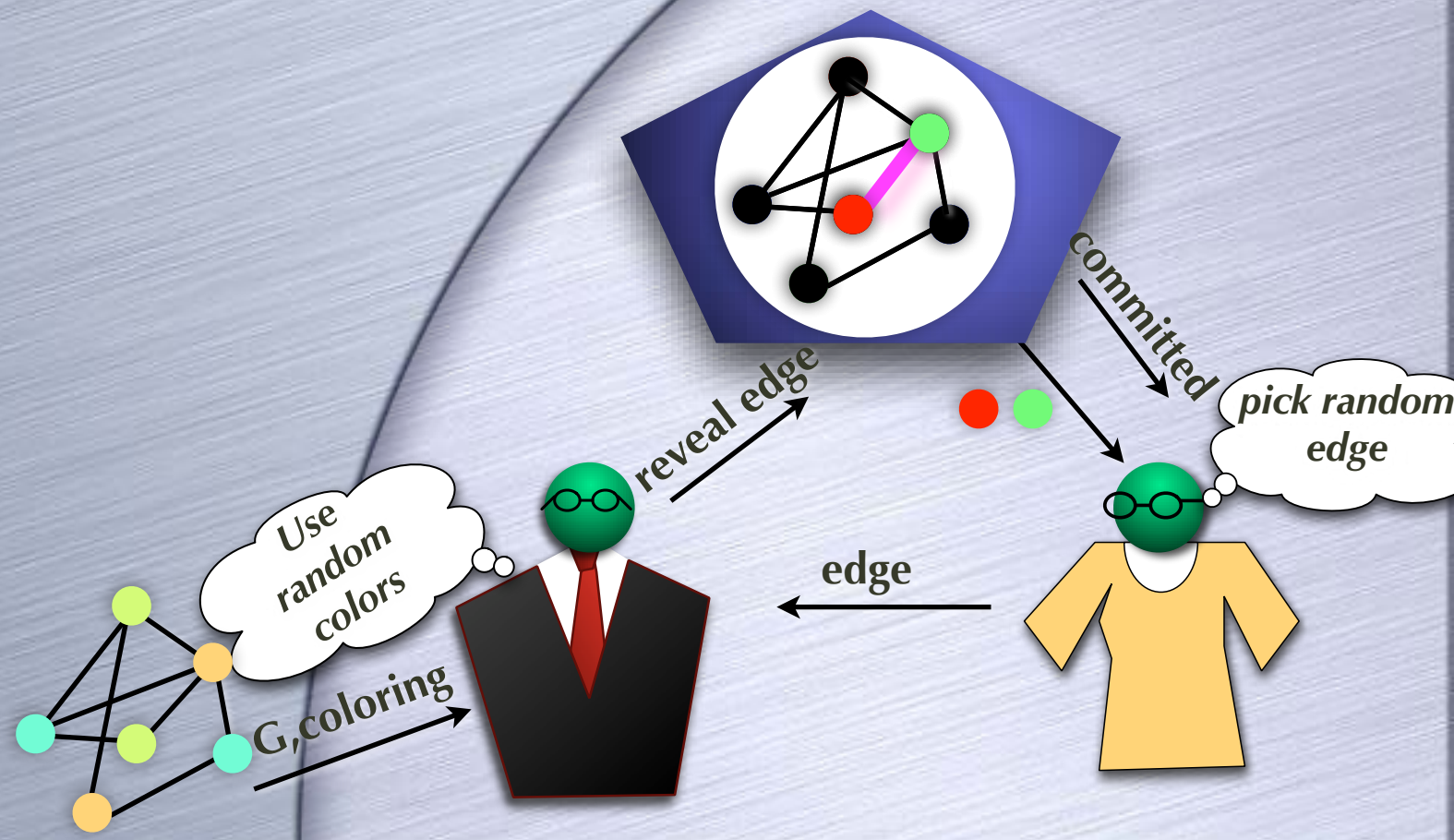

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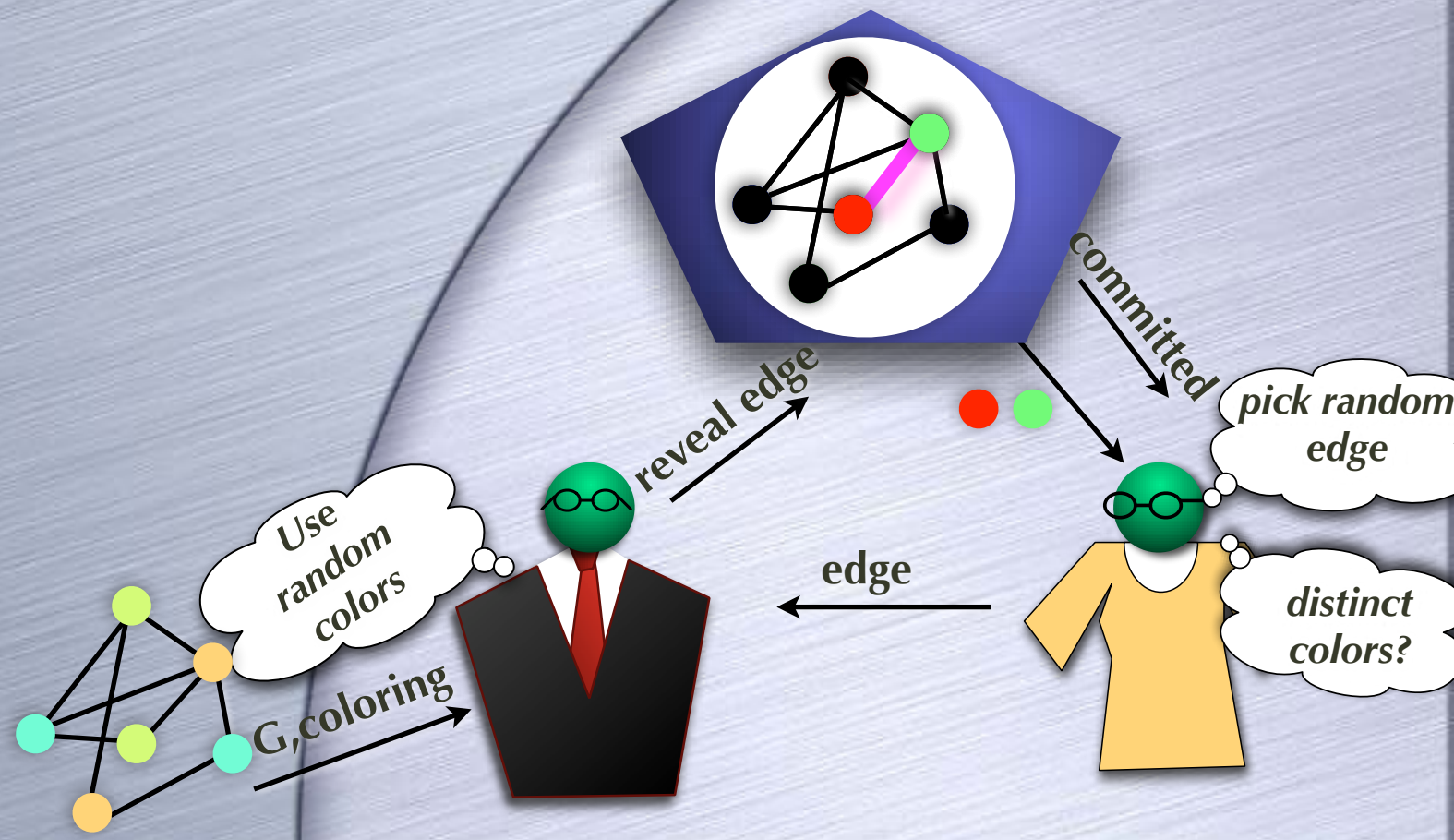

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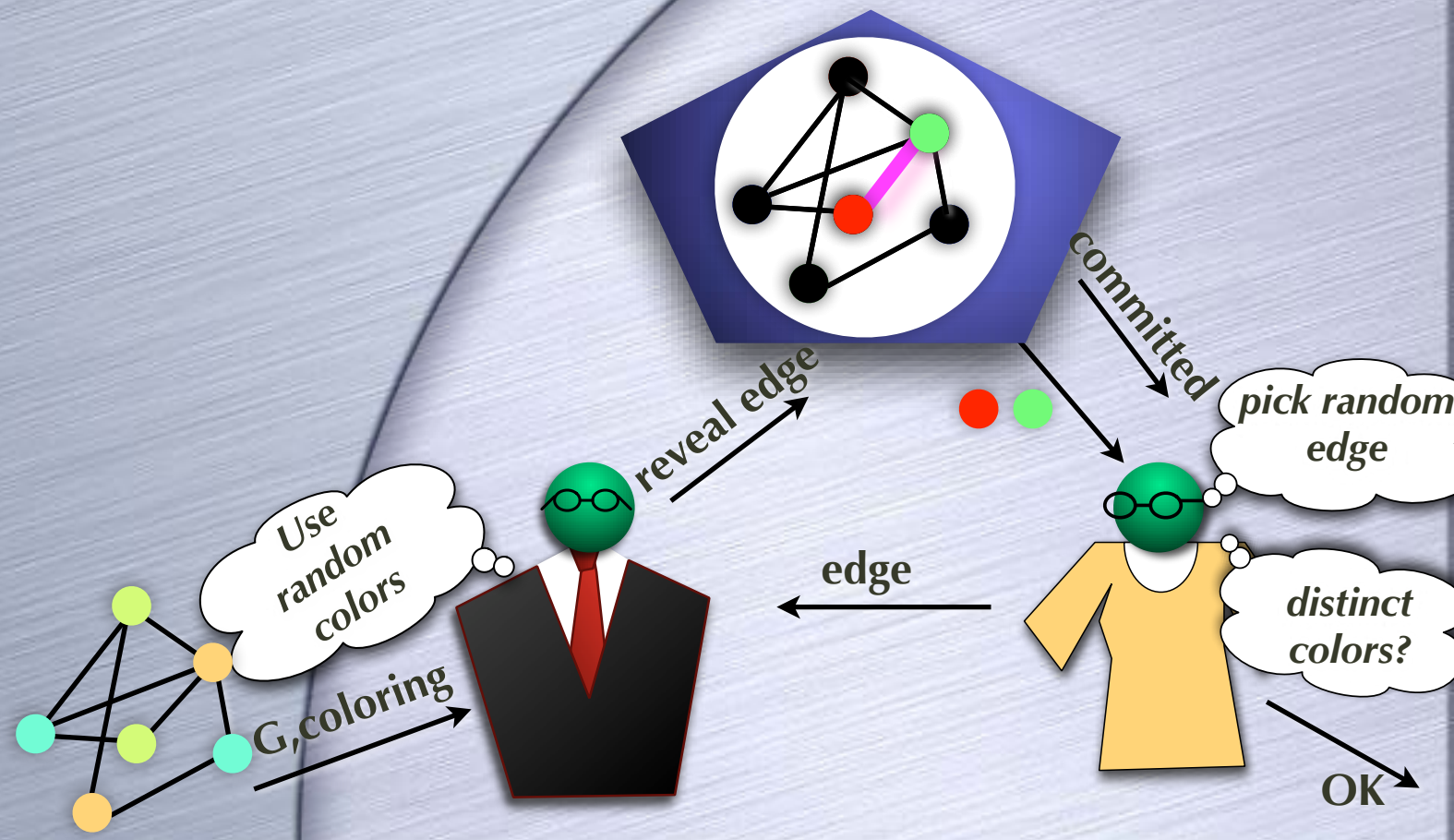

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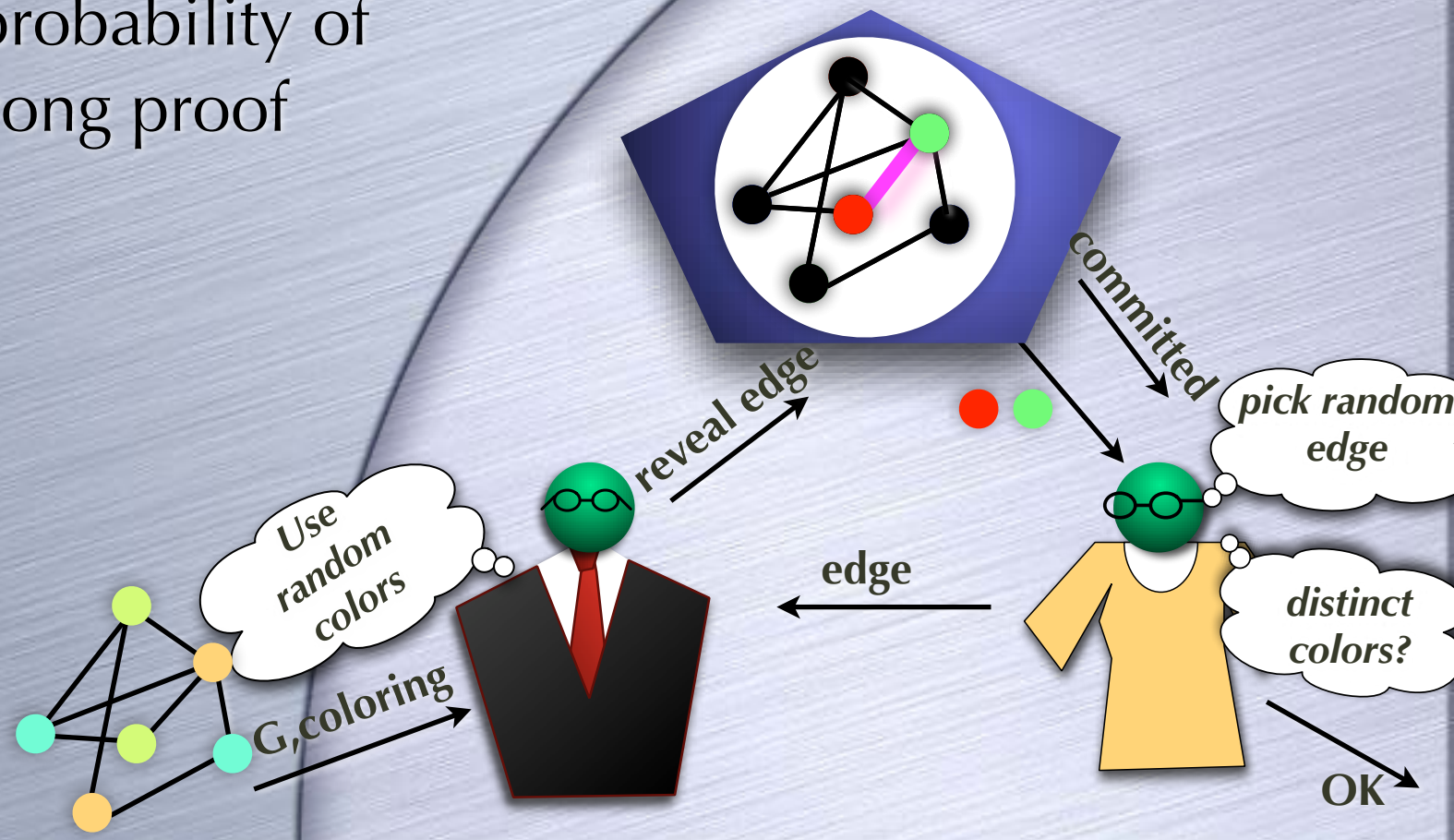

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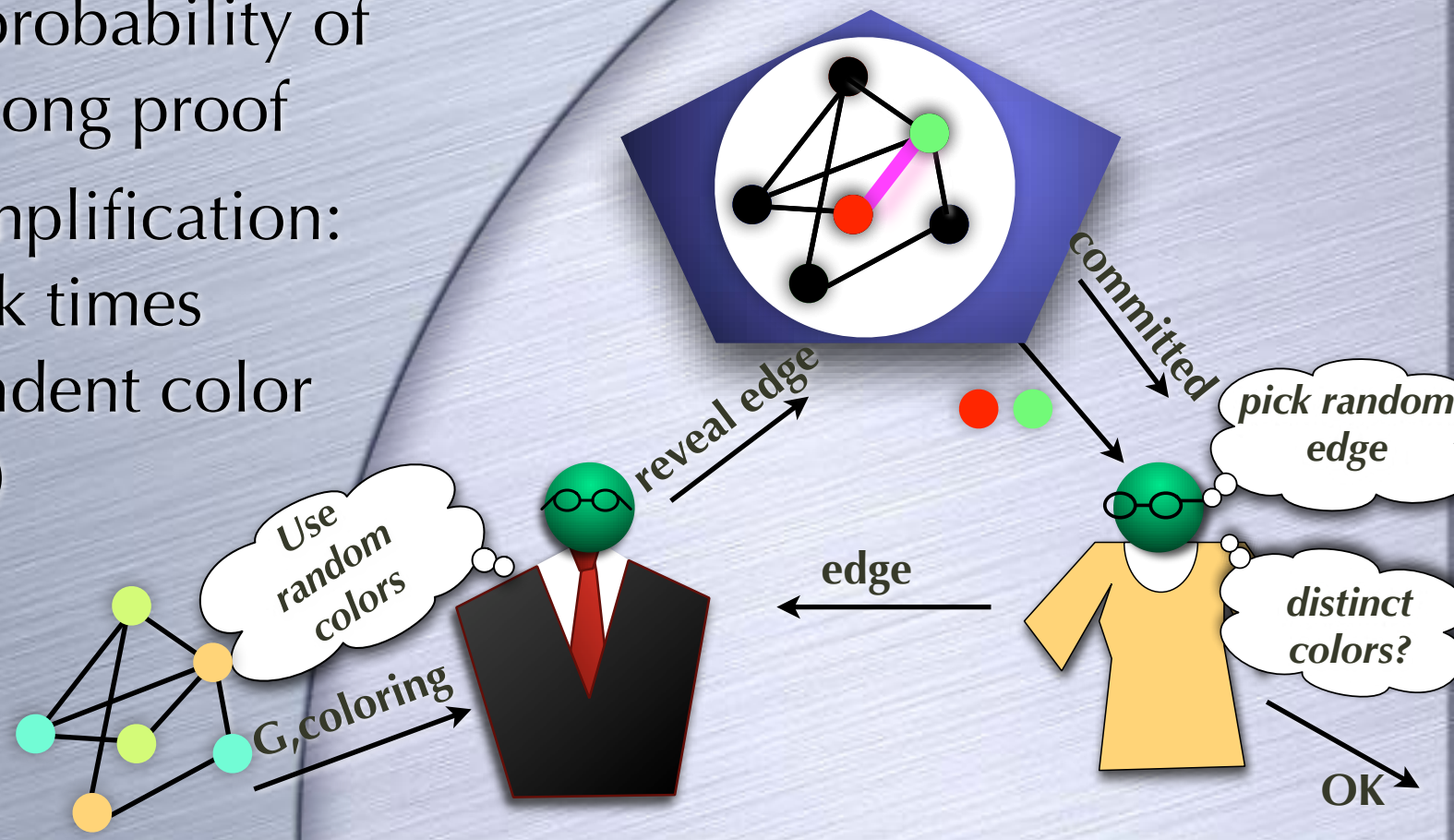

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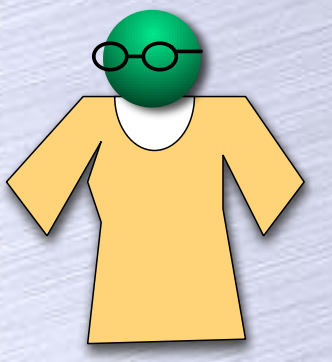

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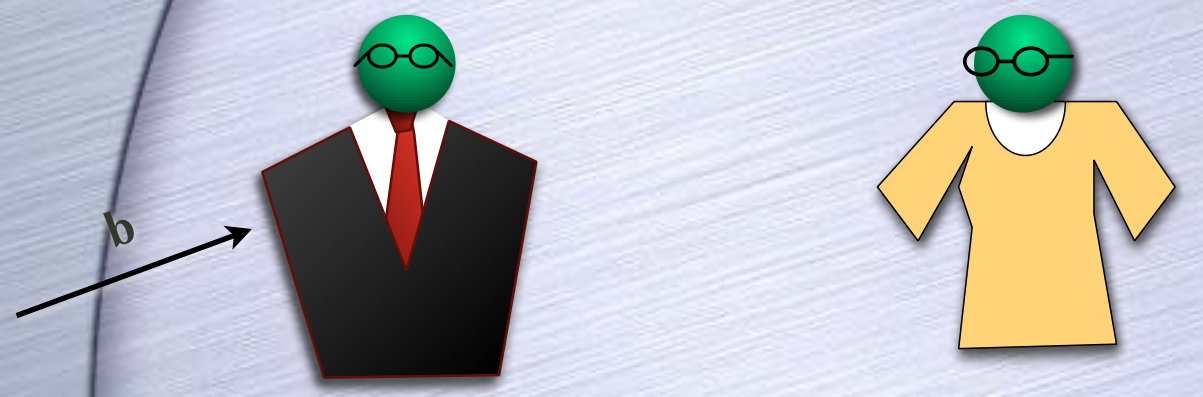

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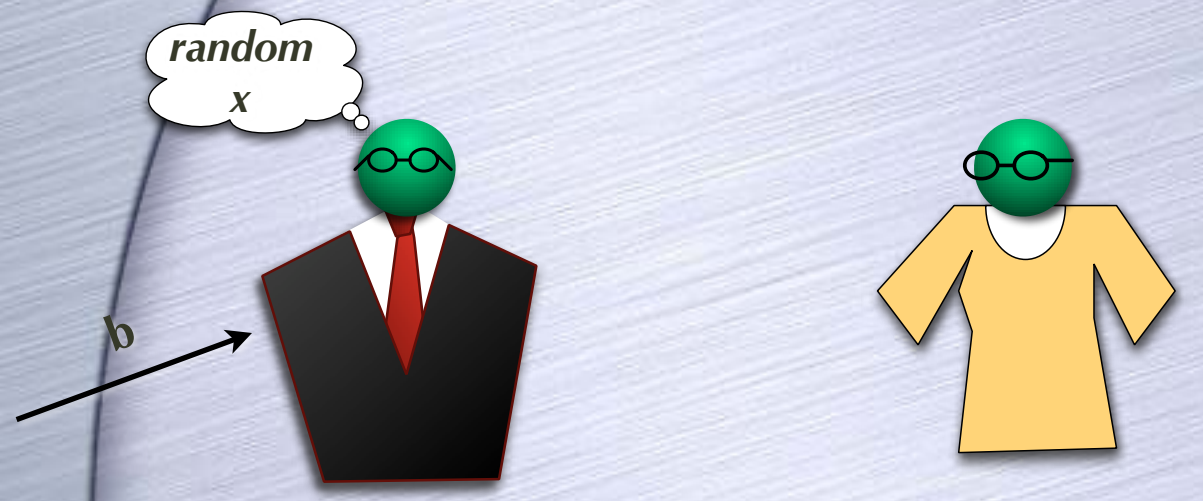

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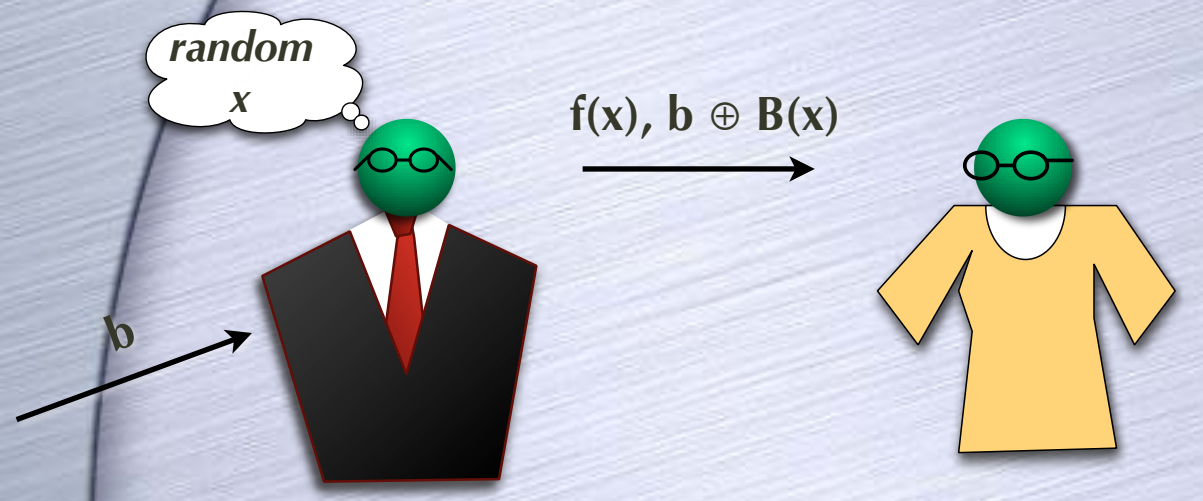

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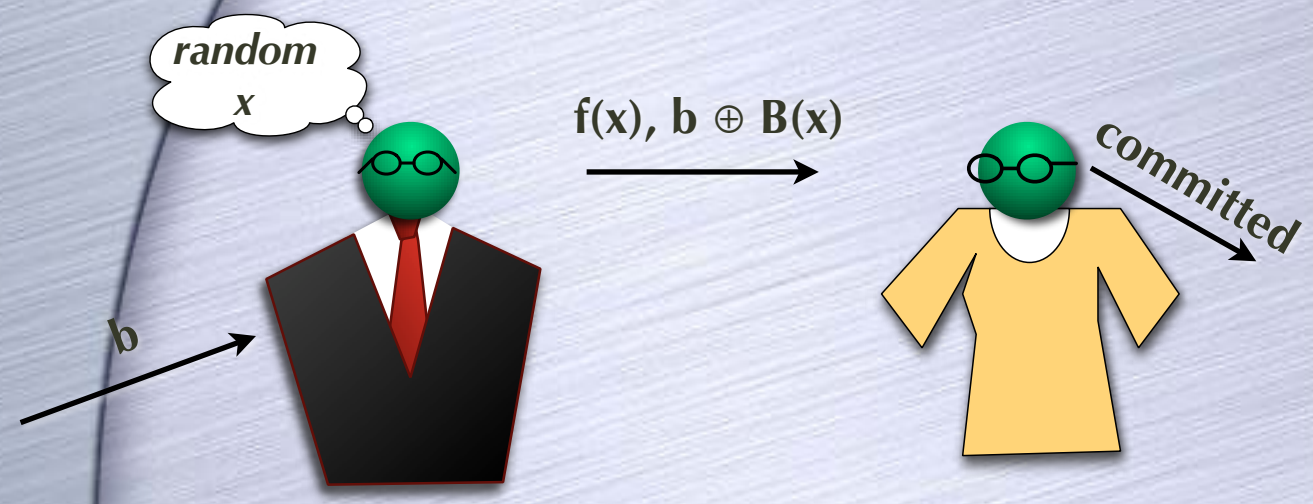

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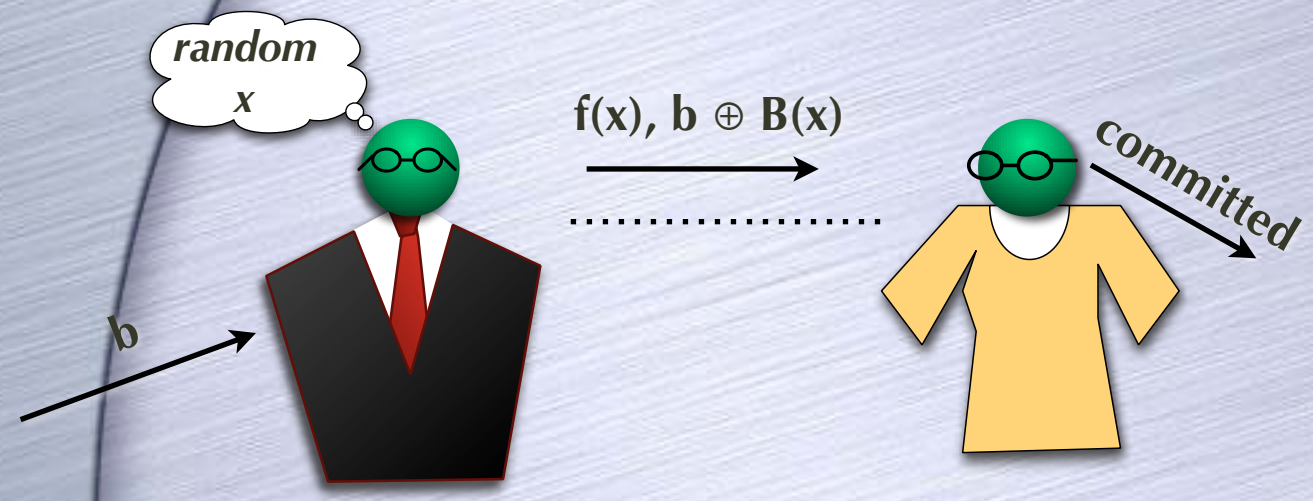

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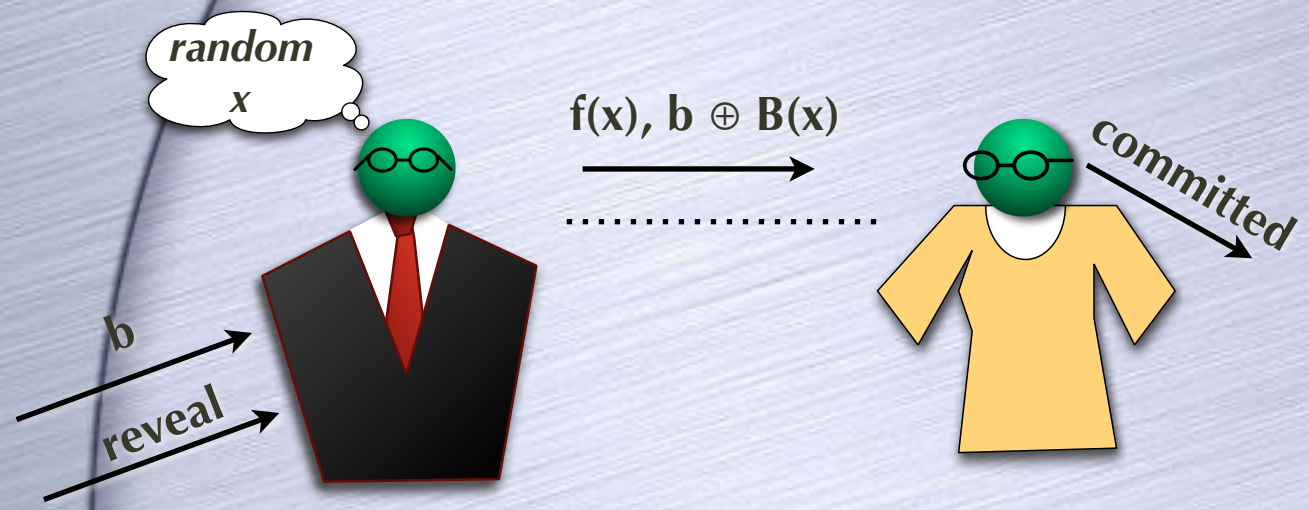

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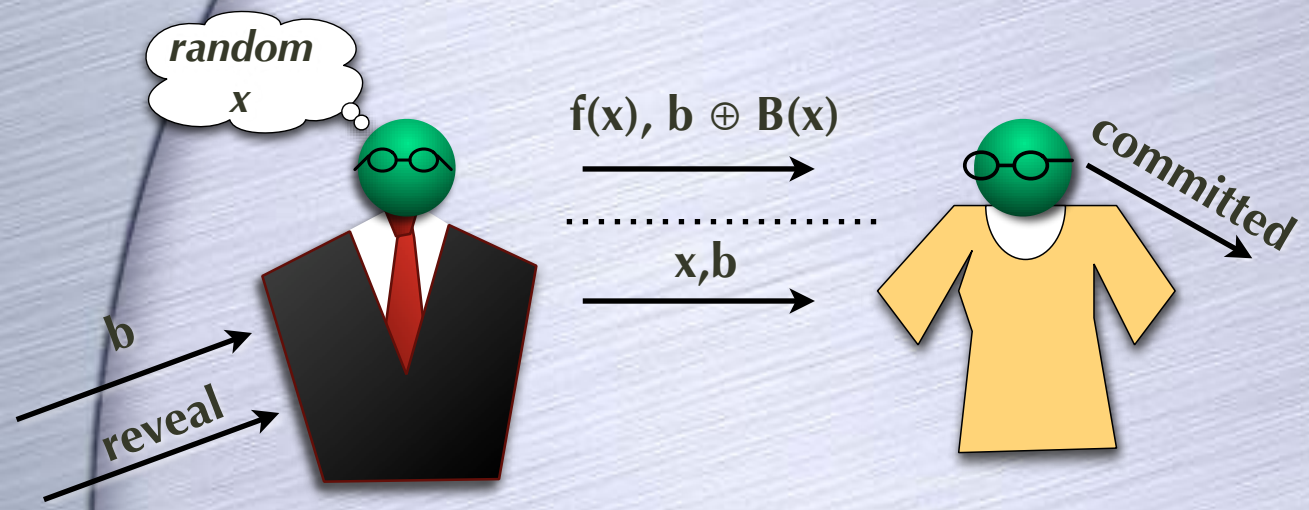

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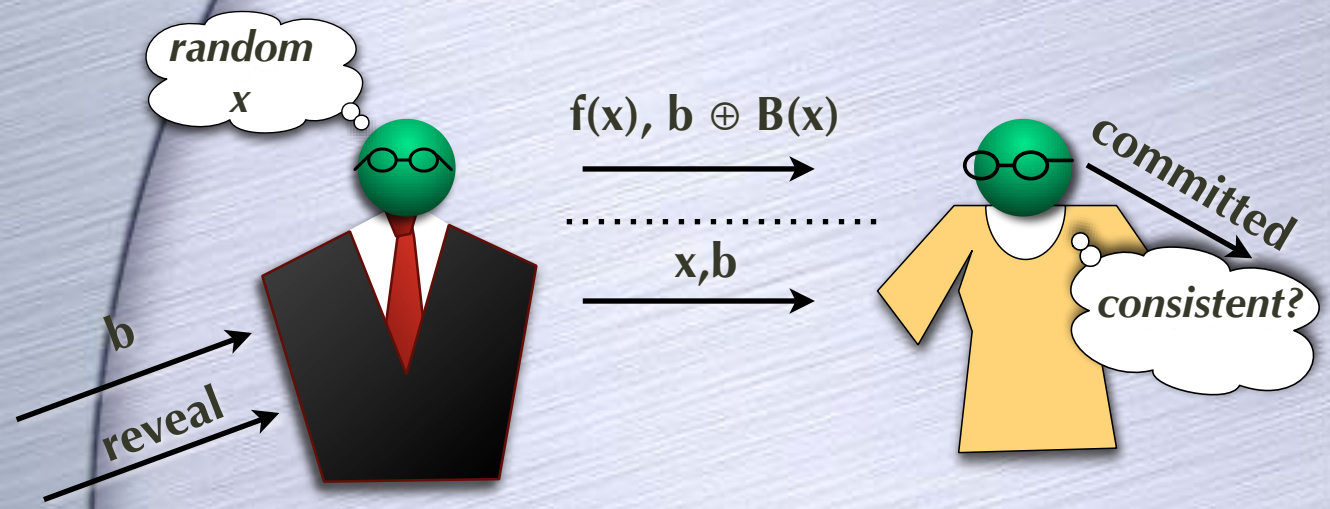

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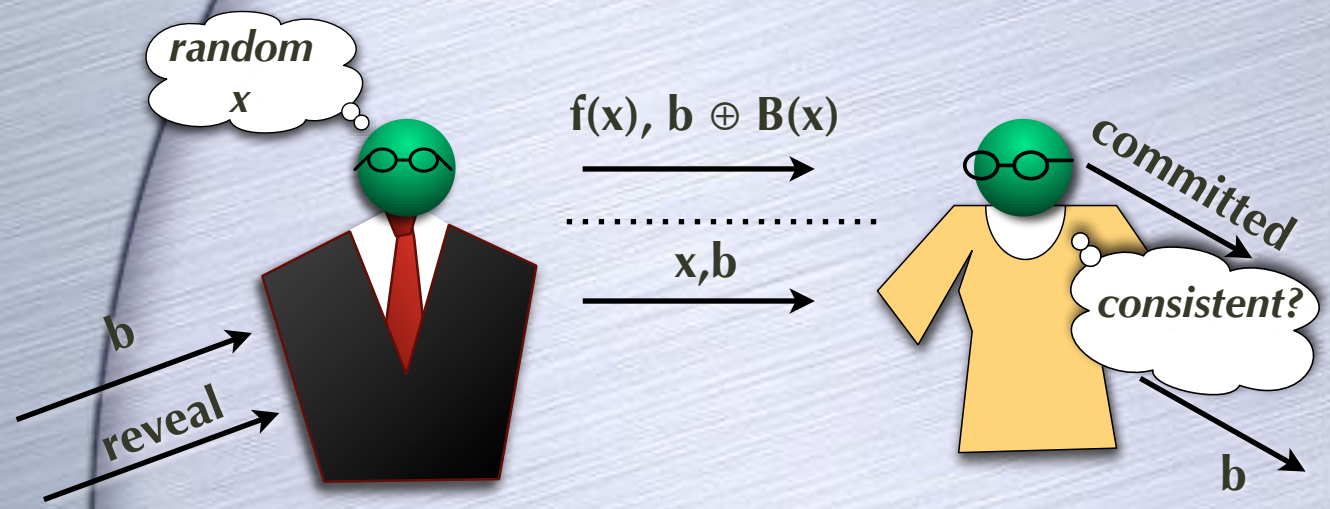

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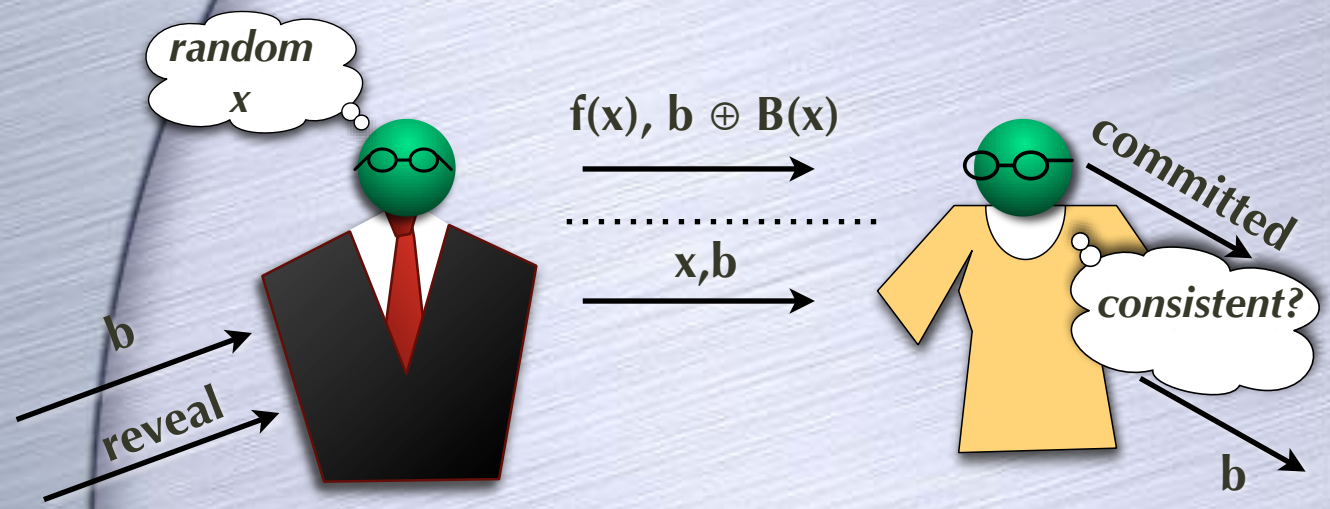

# A Commitment Protocol

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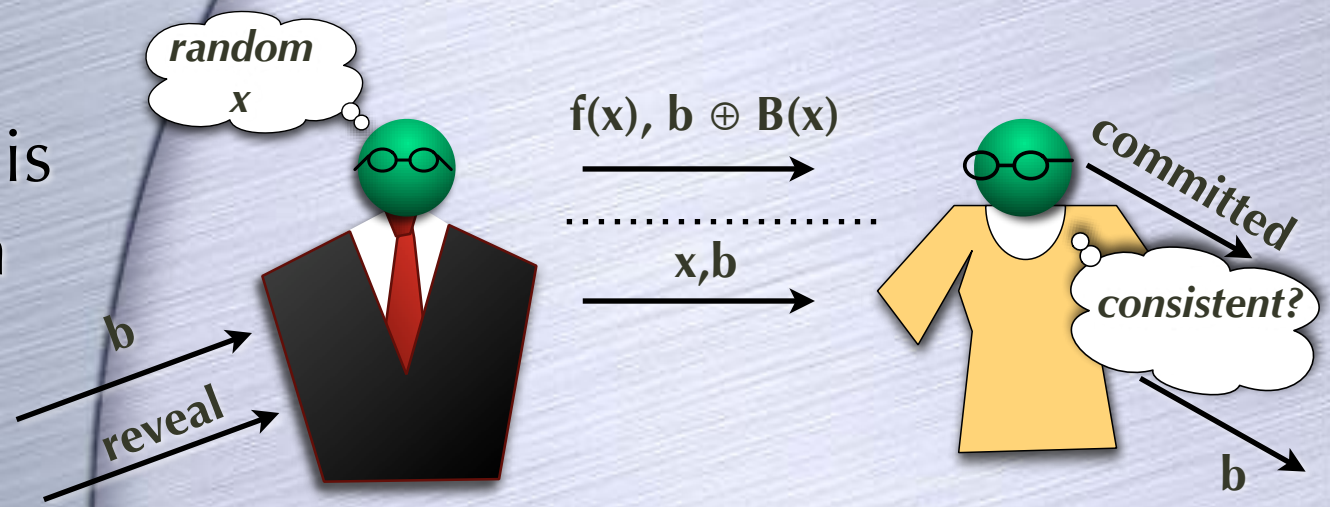
- Perfectly binding because f is a permutation
- Hiding because B(x) is pseudorandom given f(x)



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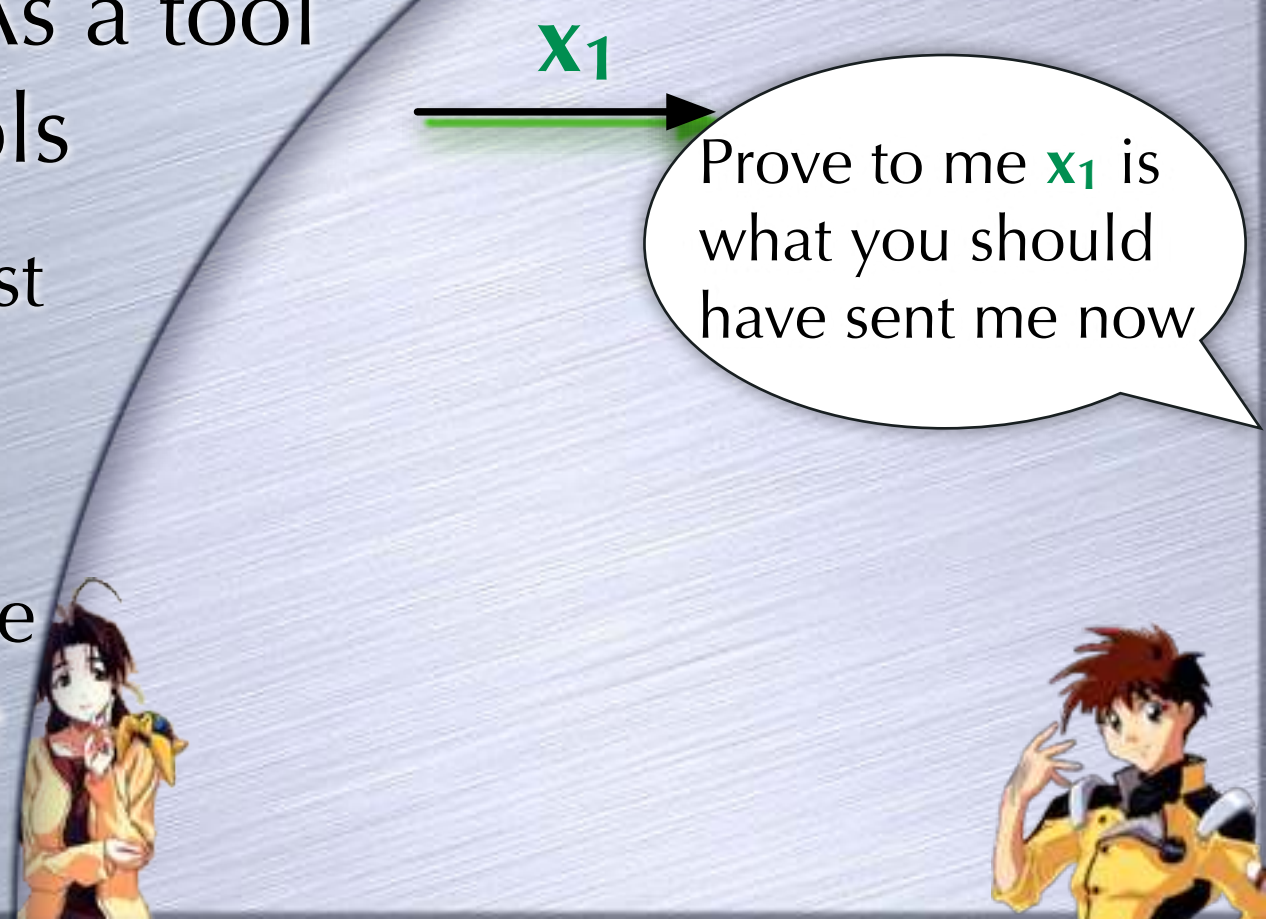

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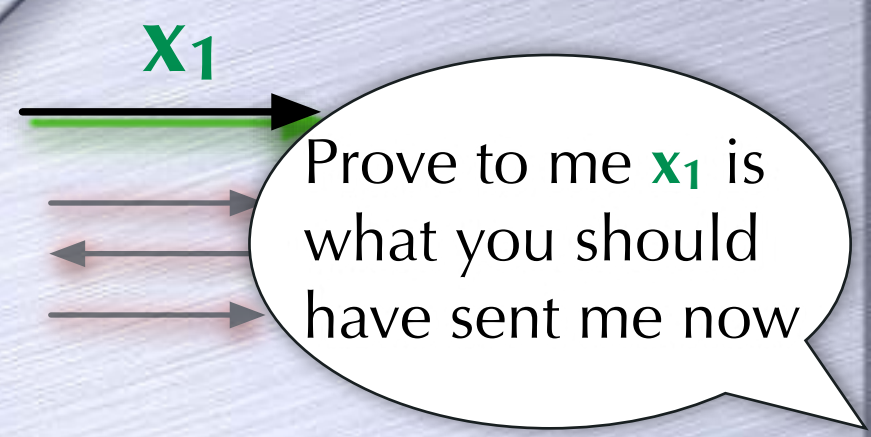


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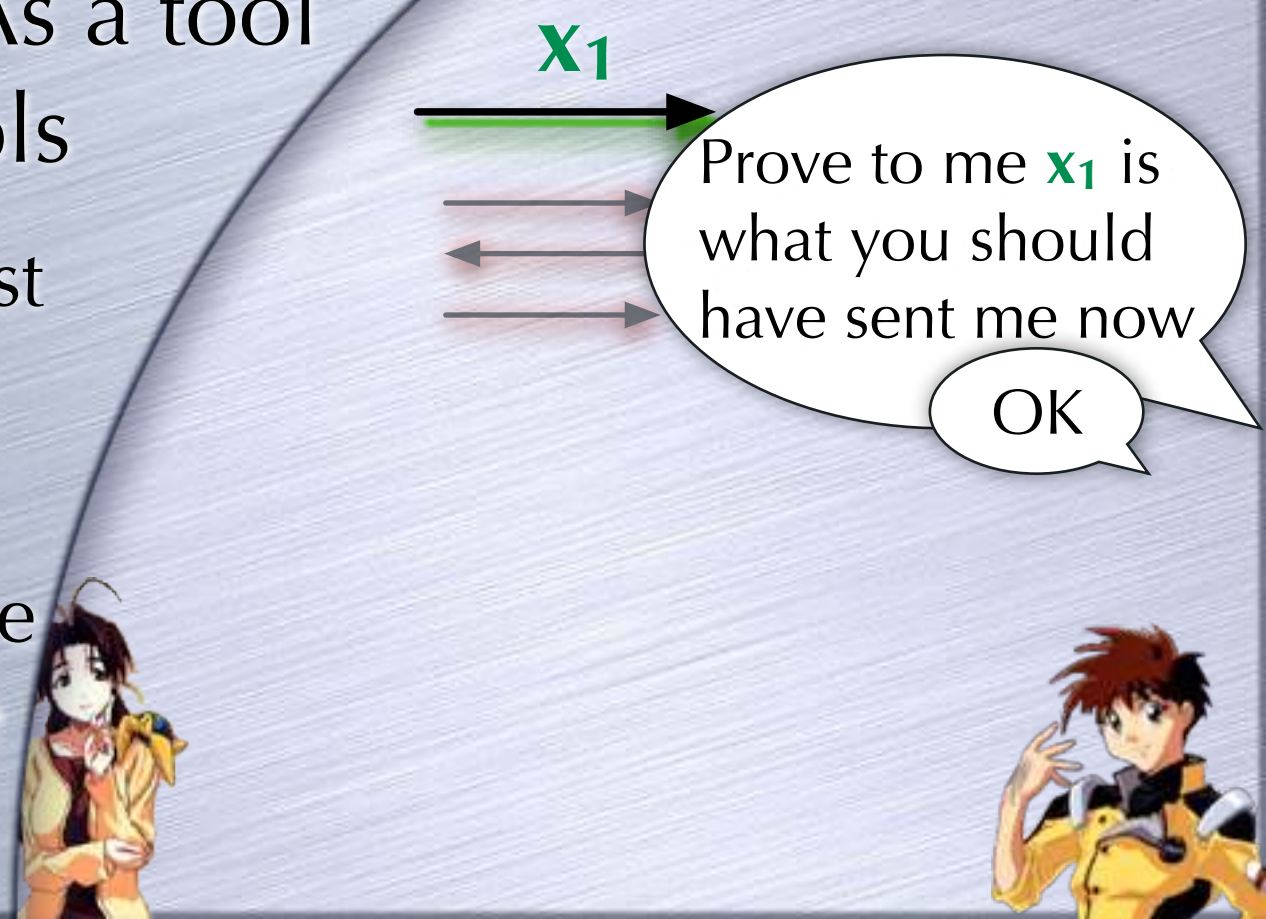

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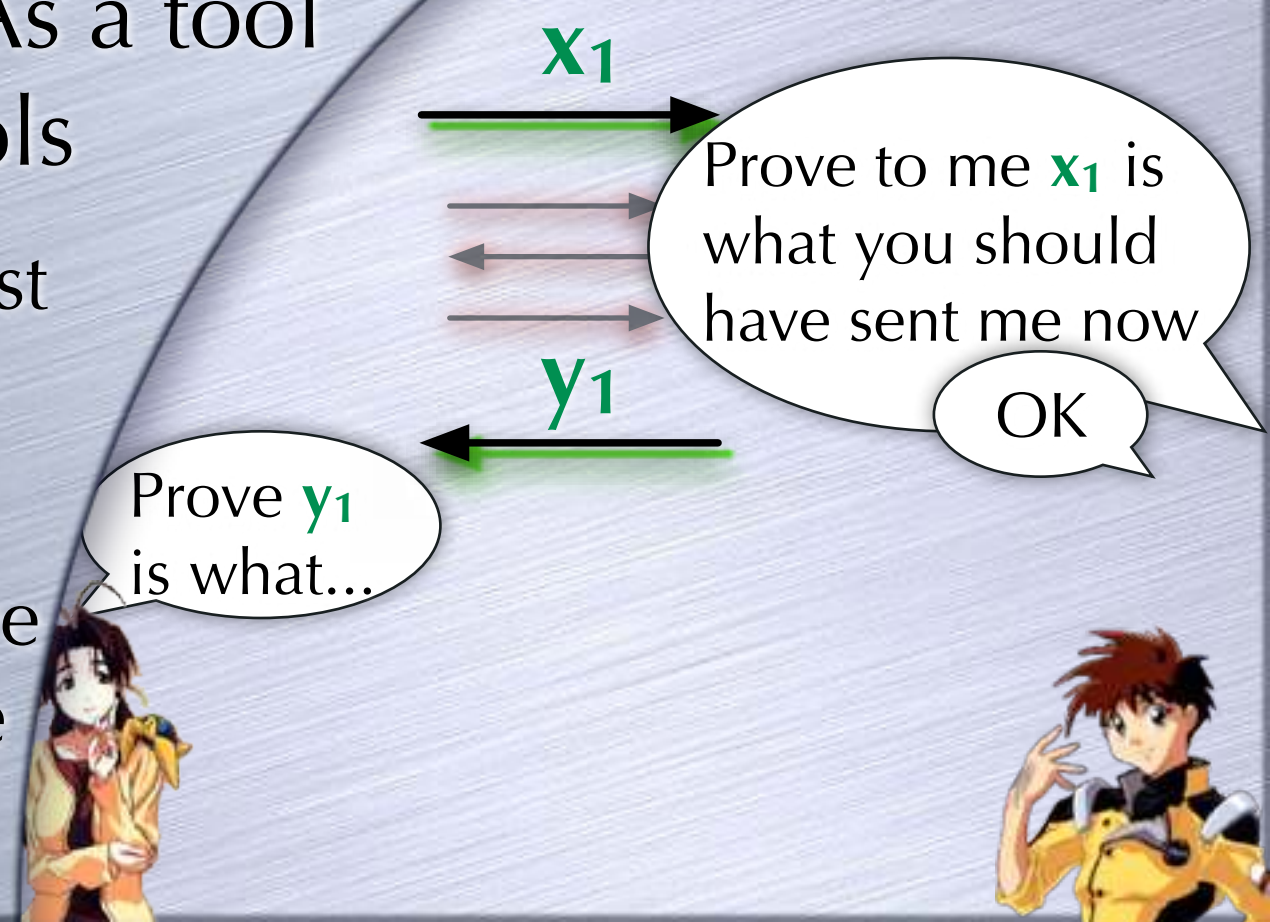

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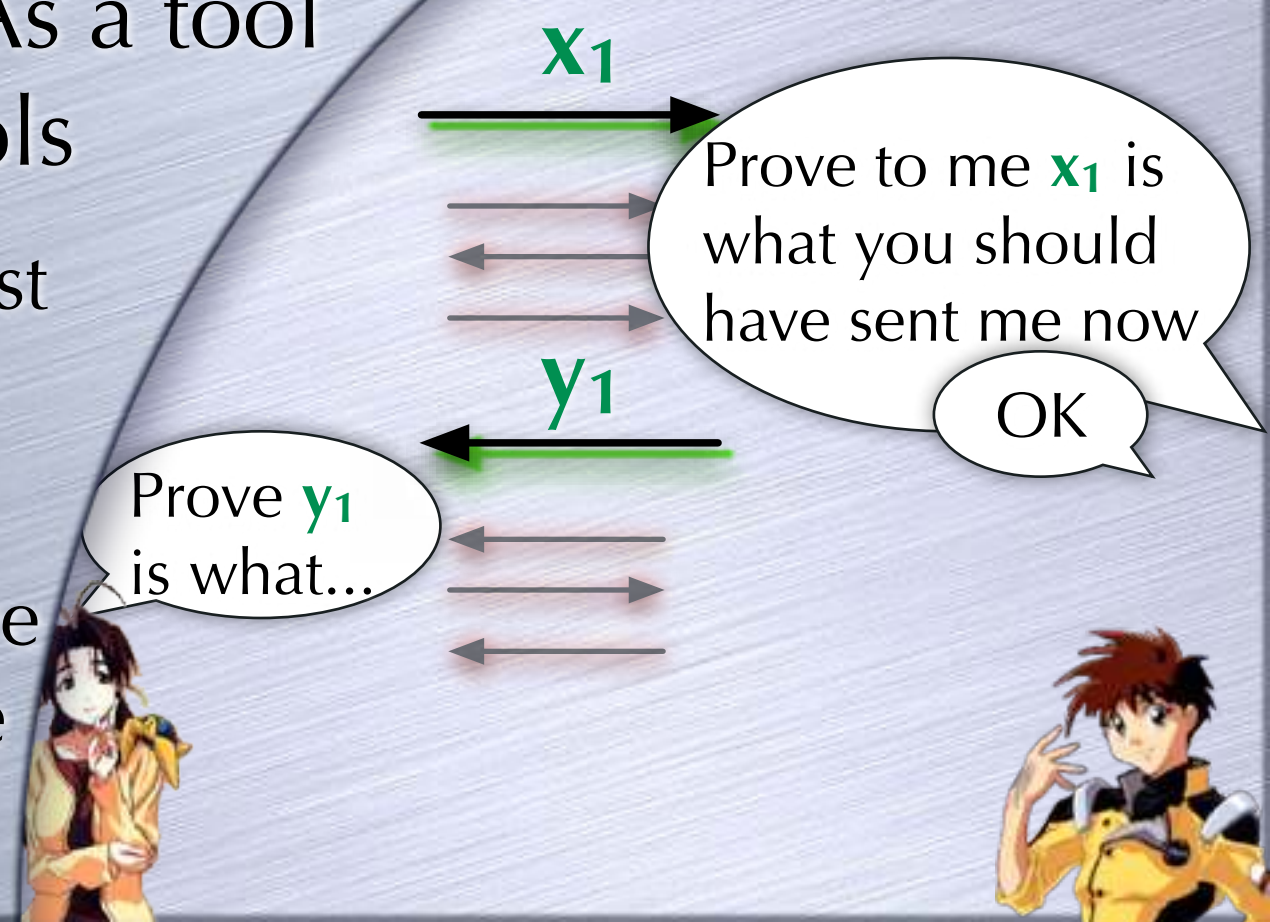

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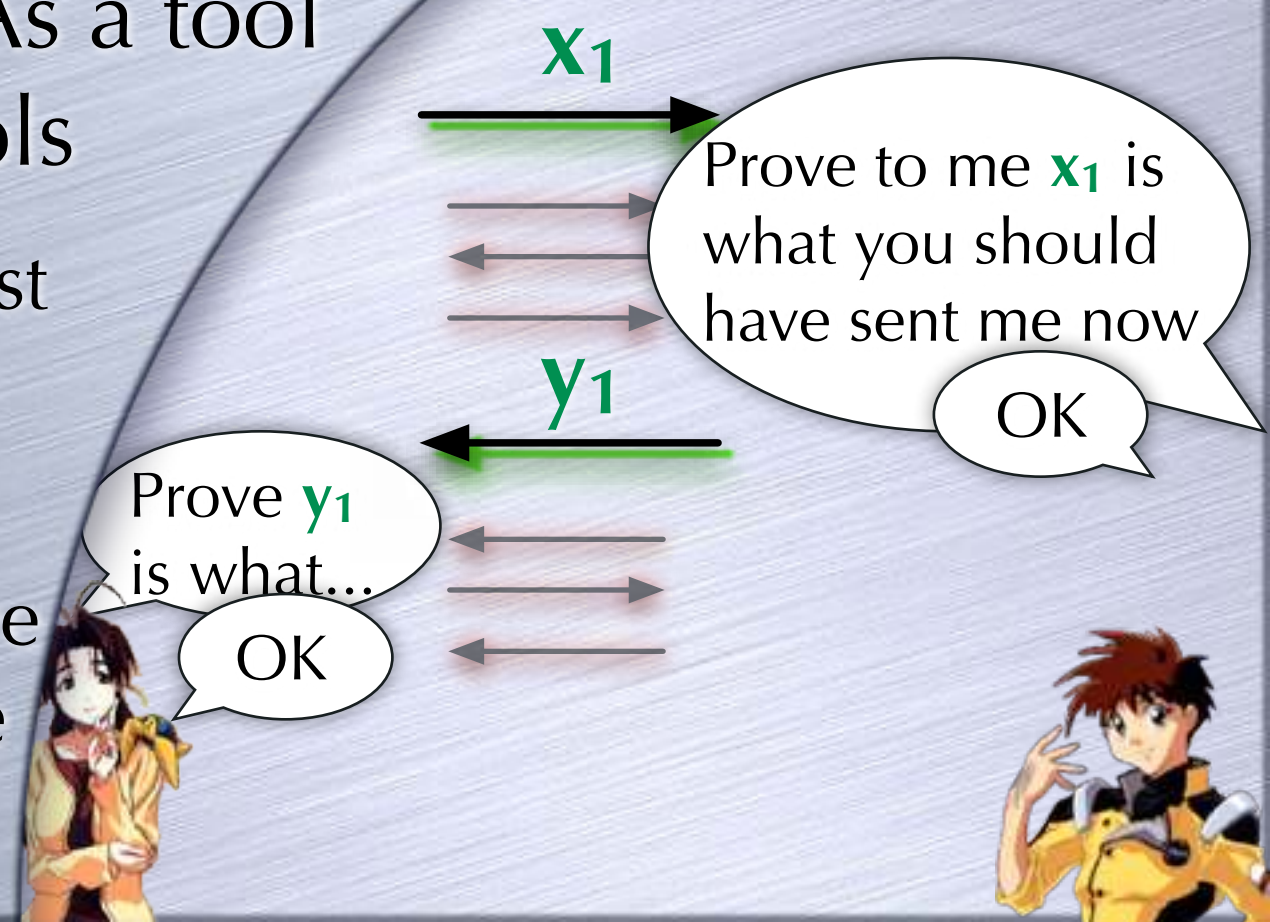

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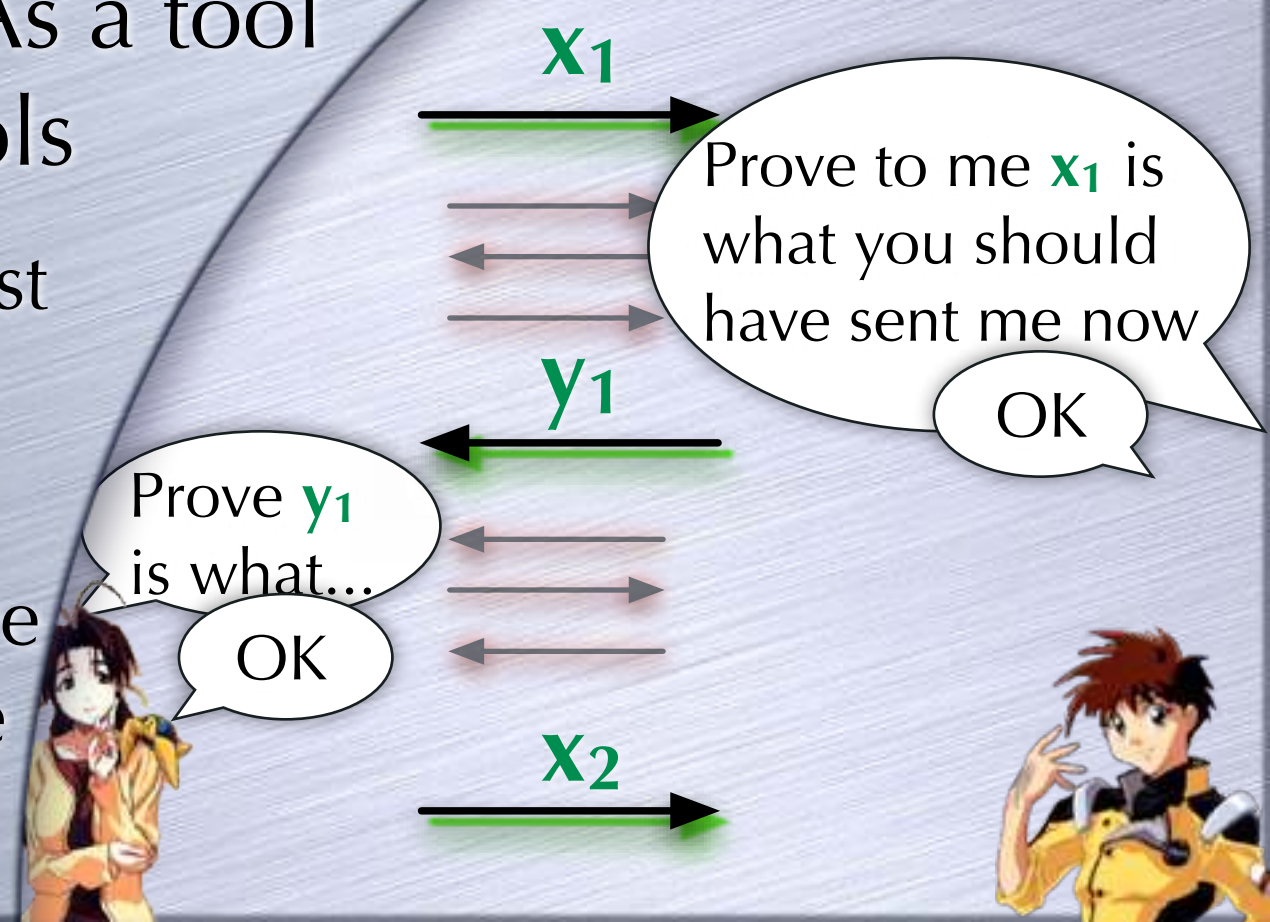

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