Secure Multi-Party Computation

Lecture 17

GMW & BGW Protocols

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  - Going from passive to active security

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- Idea: Evaluate a circuit with wire values secured using (linear) **secret-sharing**

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- Will write $[s]_1$ and $[s]_2$ to denote shares of s

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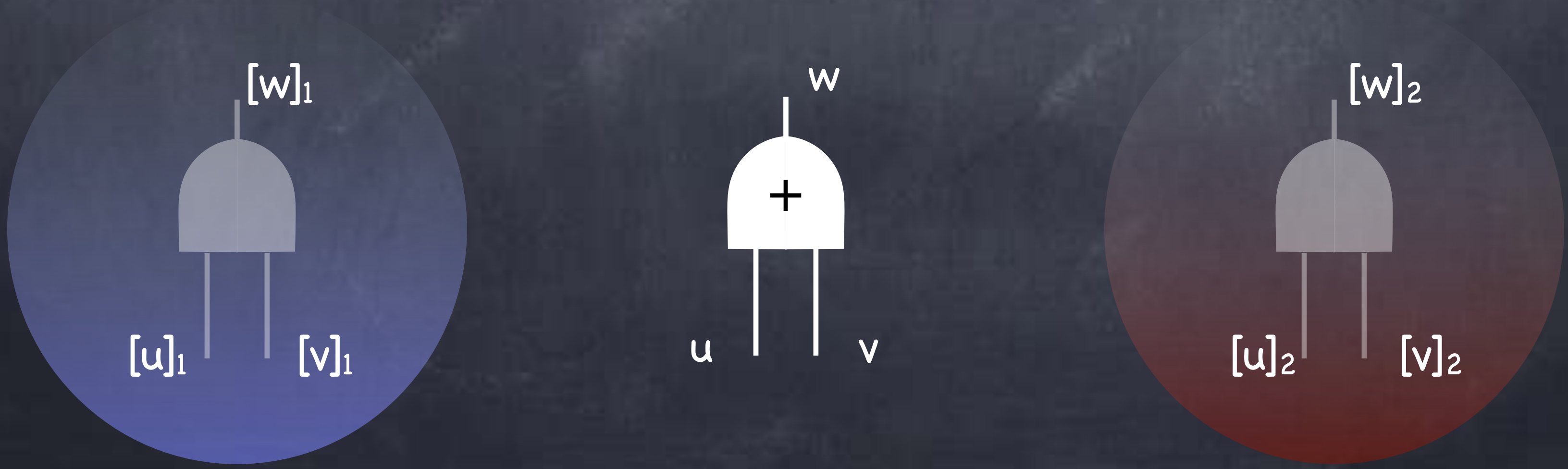

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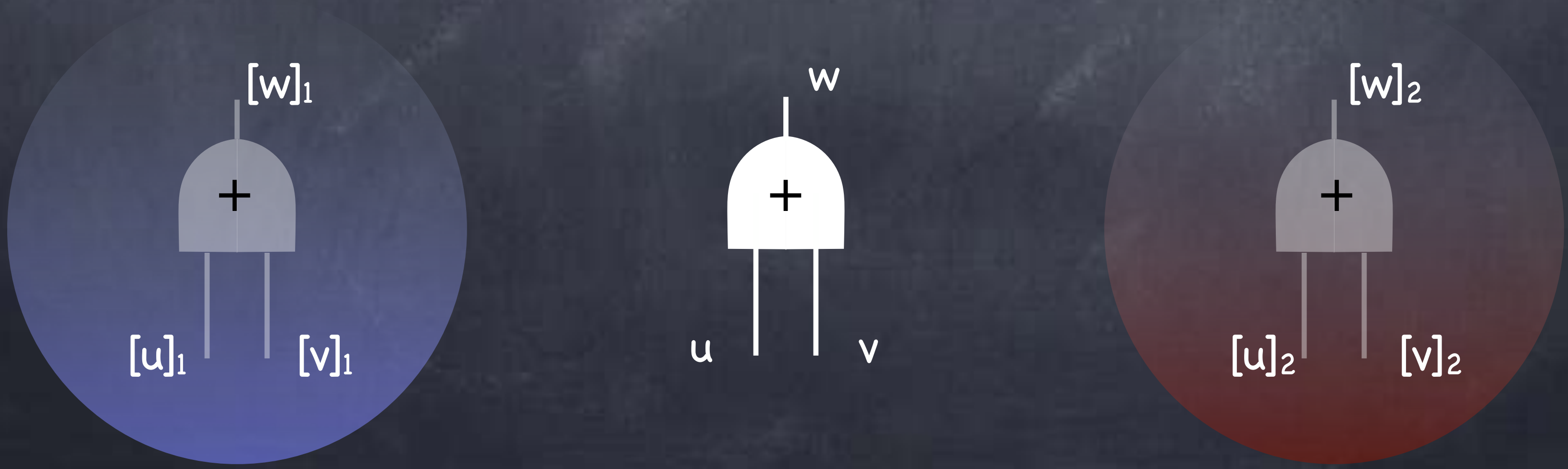

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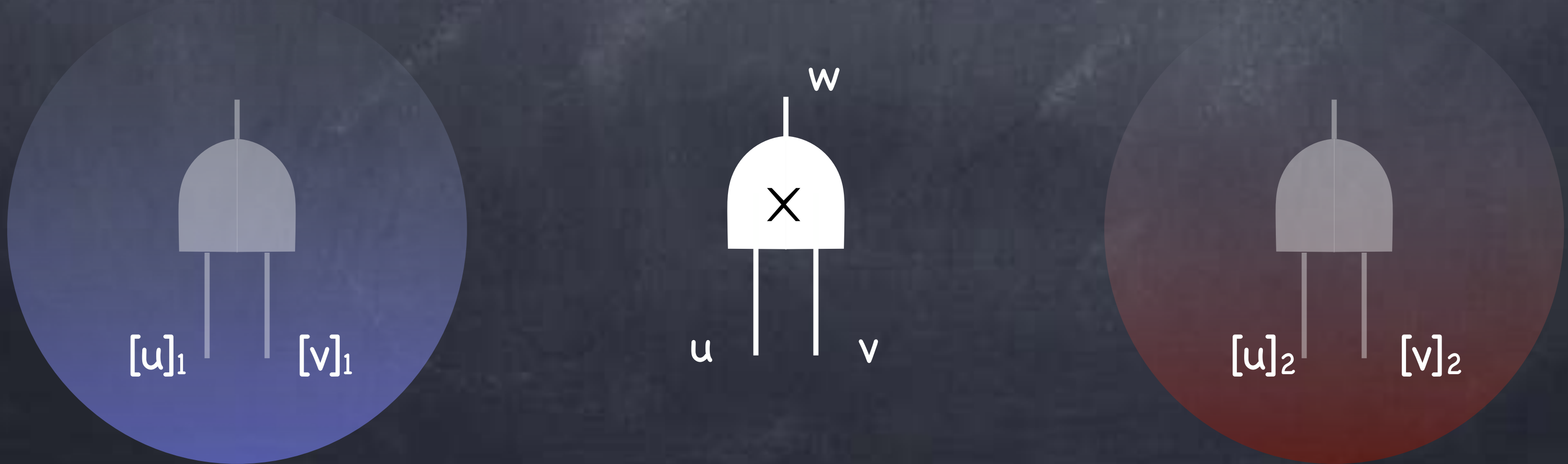

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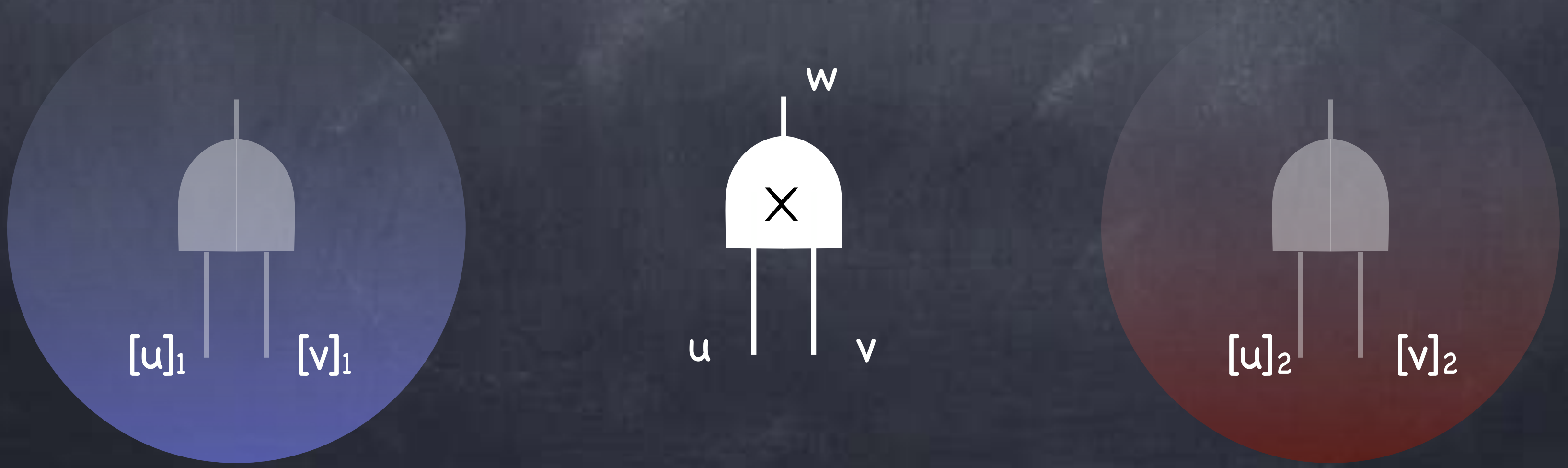

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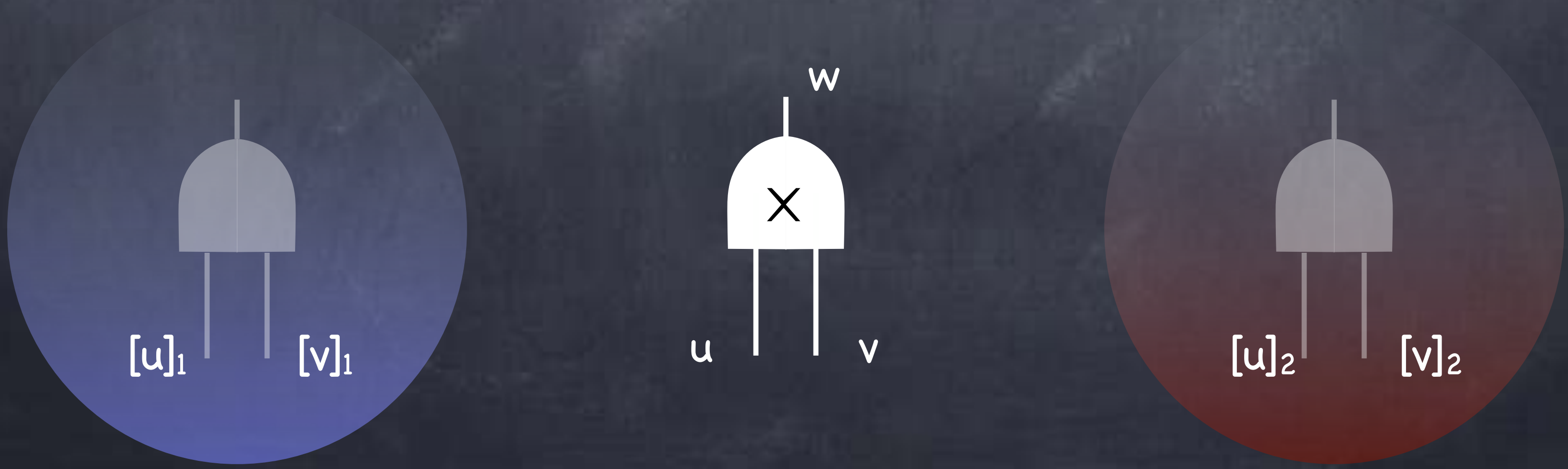

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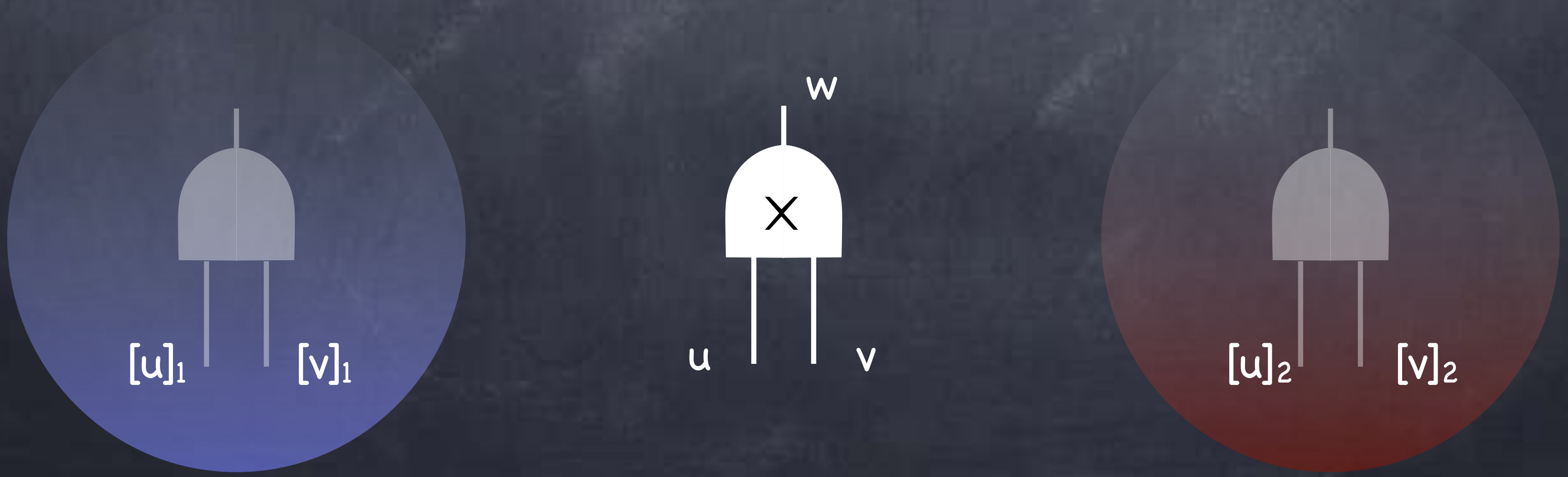

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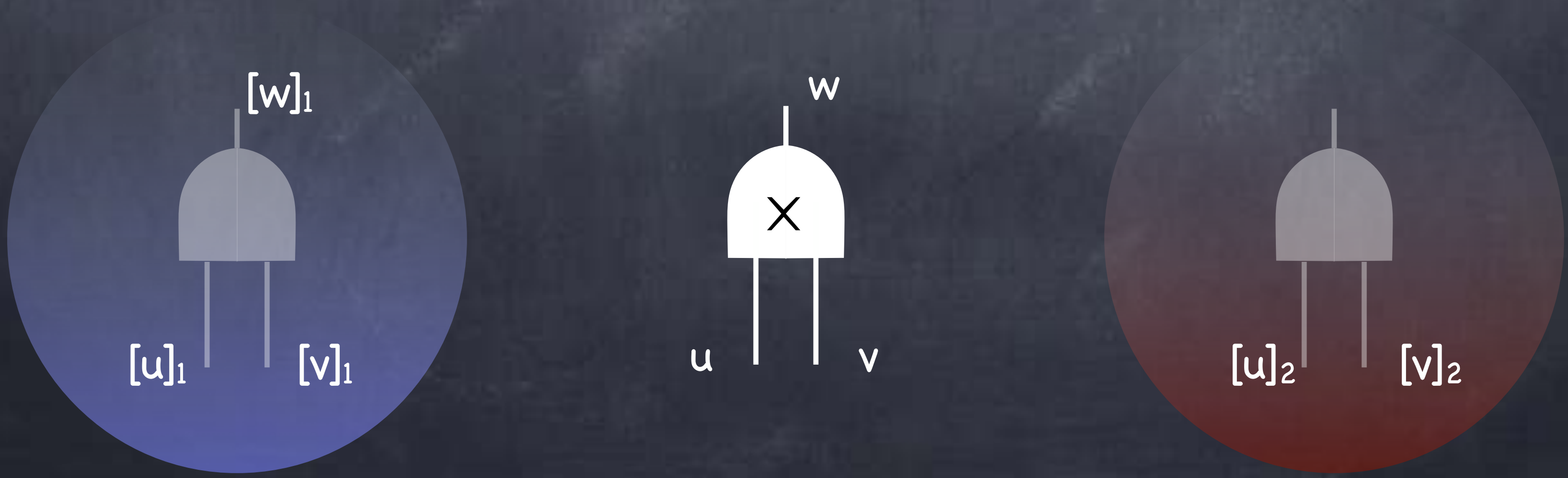

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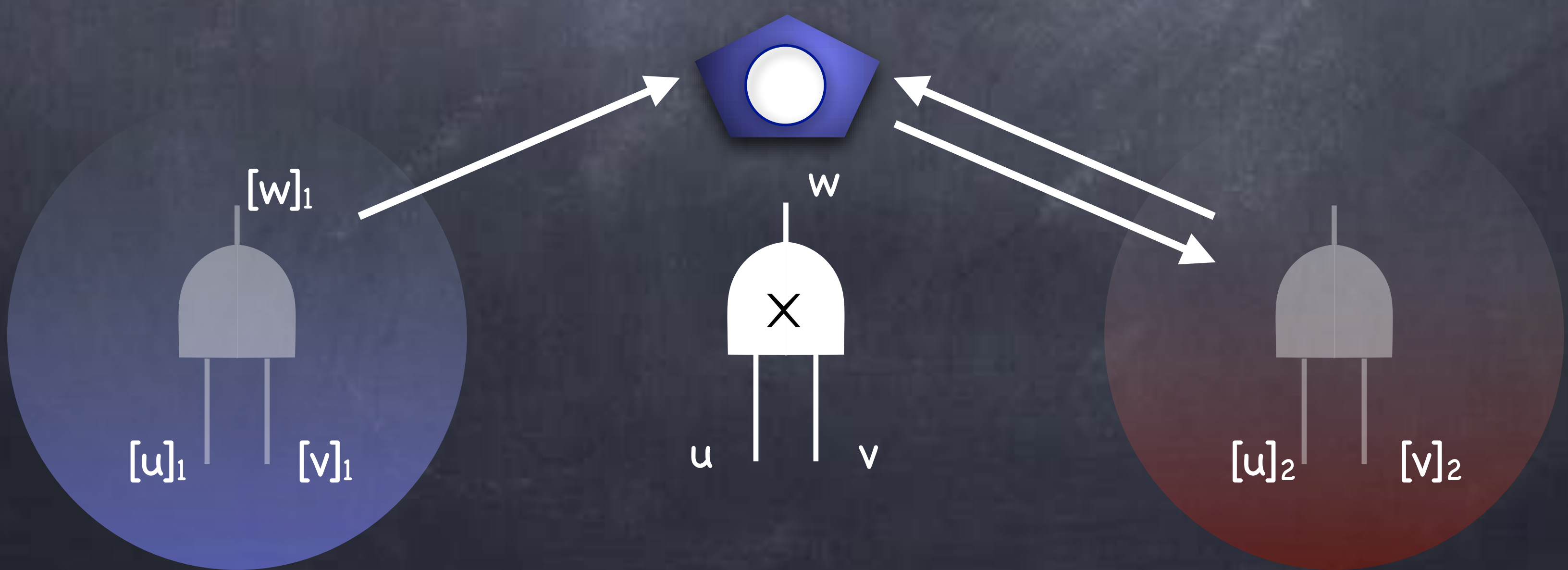

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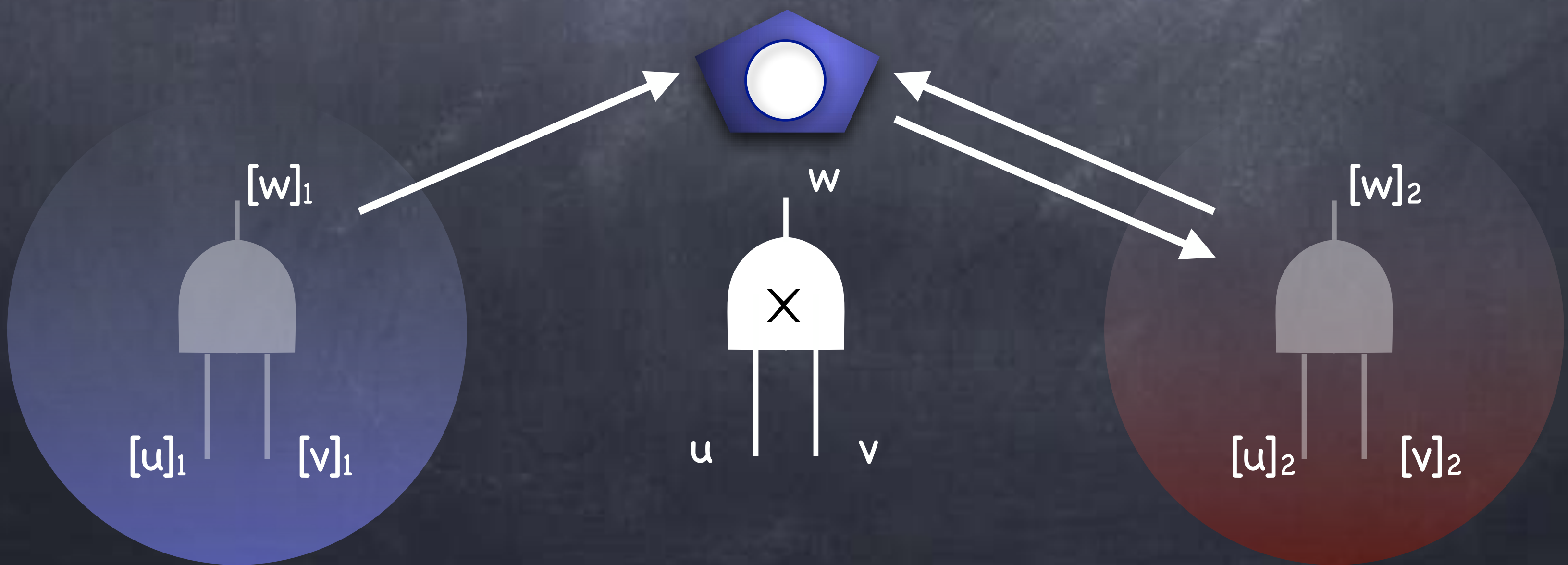

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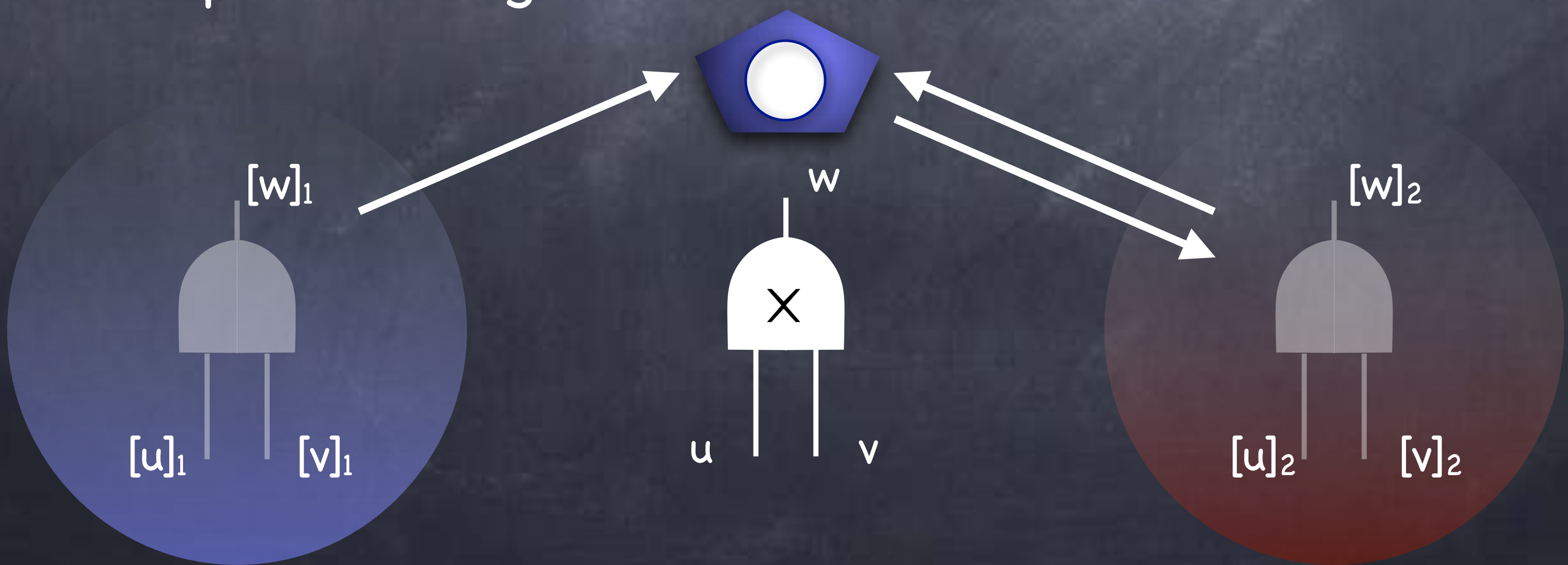
    - Bob's input is $([u]_2,[v]_2)$. Over the binary field, this requires a single 1-out-of-4 OT.



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- Multiplication: For $w = u \times v$
  $[w]_1 +..+ [w]_m = ([u]_1 +..+ [u]_m) \times ([v]_1 +..+ [v]_m)$
  - Party i computes $[u]_i[v]_i$
  - For every pair (i,j), i≠j, Party i picks random a_{ij} and lets Party j securely compute b_{ij} s.t. $a_{ij} + b_{ij} = [u]_i[v]_j$ using the naive protocol (a single 1-out-of-2 OT)
  - Party i sets $[w]_i = [u]_i[v]_i + \Sigma_j (a_{ij} + b_{ji})$

GMW: with active corruption

- Original GMW approach: Use **Zero Knowledge proofs** (next time) to force the parties to run the protocol honestly
  - Needs (passive-secure) OT to be implemented using a protocol
- Alternate constructions give information-theoretic reduction to OT, starting from passive-secure GMW
  - Recent approach: pre-compile the circuit

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- i.e., arbitrary behavior of Alice & Bob while sharing $[u]_1 \cdot [v]_2$ correspond to them locally changing their shares $[u]_1$ and $[v]_2$

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- Also, can add deltas to all input and output wires

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  - Extension of "AMD codes"
    - e.g. encode x as a vector (x, r, xr) where r is random from a large field. Additive attacks (without knowing r) detected unless $(x+\delta_1)(r+\delta_2) = (xr+\delta_3)$: i.e., $\delta_1 \cdot r + x \cdot \delta_2 + \delta_1 \cdot \delta_2 = \delta_3$. Unlikely unless $\delta_1 = 0$.

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- Up Next: Adversary can corrupt any set of less than t parties out of m parties (e.g., t = n/2, t=n/3)
  - Then, can get (UC) security just from secure communication channels
  - Bonus (omitted): Can ask for guaranteed output delivery

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- Degree of $\pi \leq n-1$: $\pi(0)$ reconstructible from n shares

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- Can't tolerate active corruption of n/3 parties (even for "broadcast") if guaranteed output delivery needed
- More generally, guaranteed output delivery not possible if:
  - set of parties can be partitioned into three sets, $S_1 \cup S_2 \cup S_3$ such that S_1, S_2 (separately) may be passively corrupt, and S_3 may be actively corrupt

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  - Needs $t < n/3$. (Or $t < n/2$, but using a secure broadcast channel)

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