Symmetric-Key Encryption: constructions

Lecture 4

PRG, Stream Cipher

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- Today: Constructing SKE from Pseudorandomness
- Next time: Pseudorandomness ← One-Way Permutations

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- Theoretical Constructions: Security relies on certain computational hardness assumptions related to simple functions

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$| \Pr_{y \leftarrow PRG}[A(y)=0] - \Pr_{y \leftarrow rand}[A(y)=0] |$ is negligible for all PPT A

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- If X_k, X'_k are short (say a single bit), $X_k \approx X'_k$ iff X_k, X'_k are statistically indistinguishable (Exercise)

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    - i.e., no PRG against unbounded adversaries

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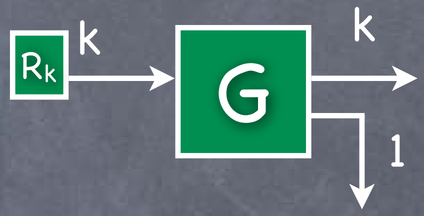

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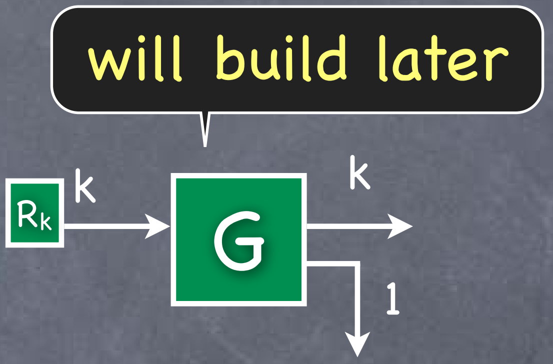

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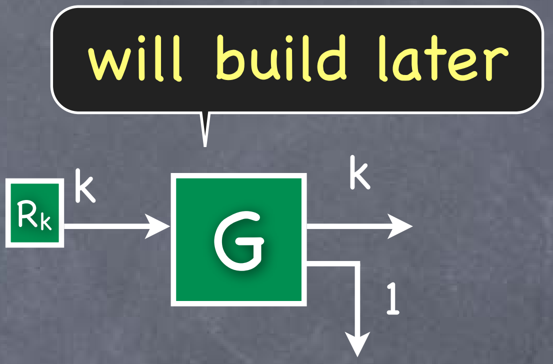

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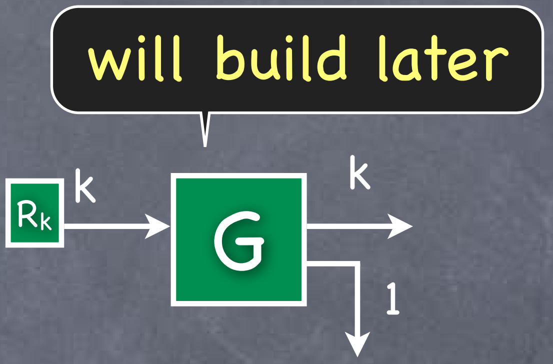

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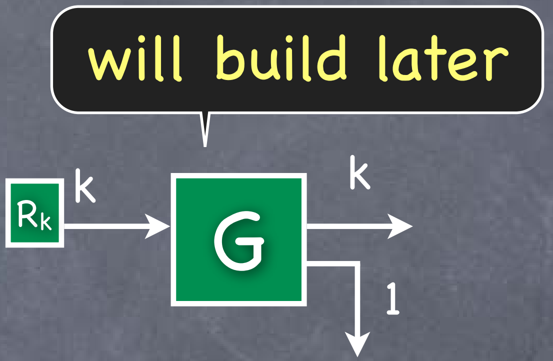

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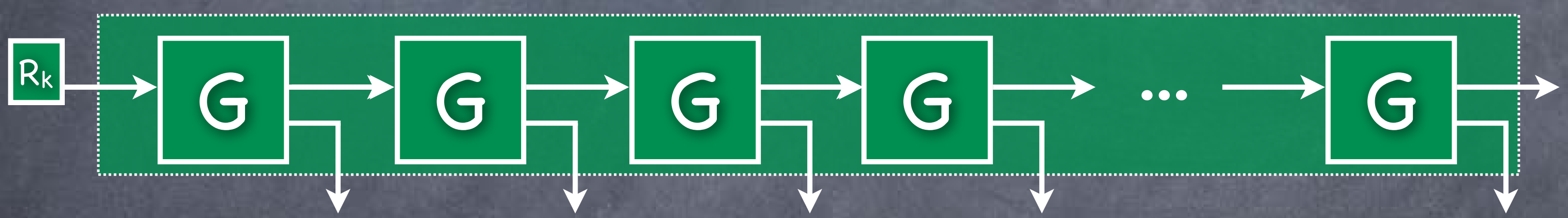

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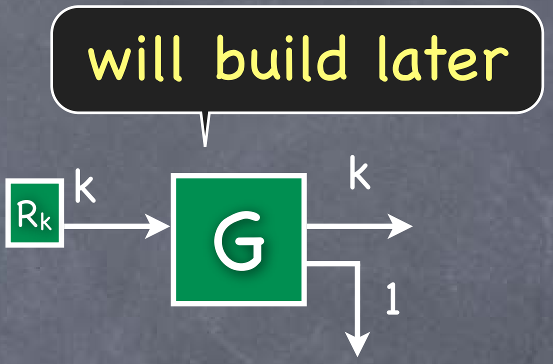

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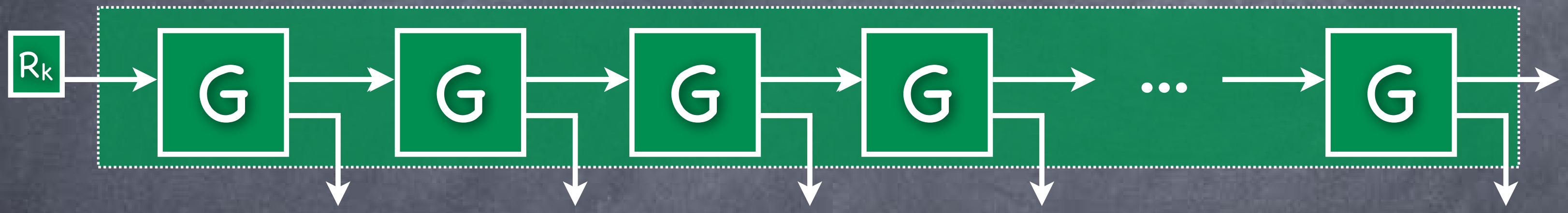

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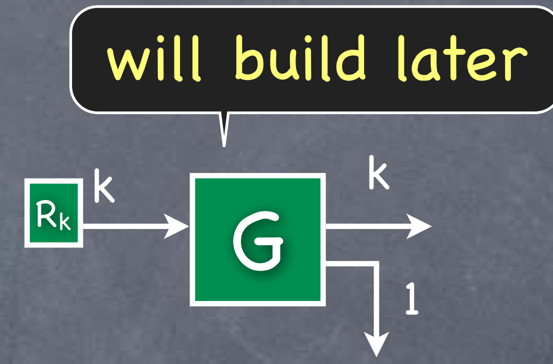

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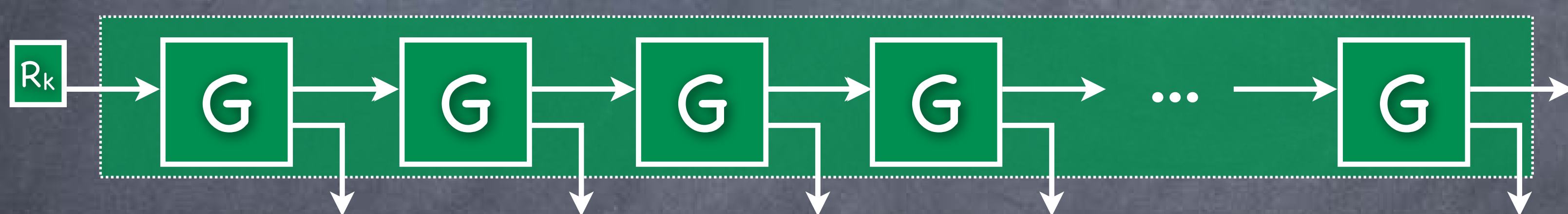

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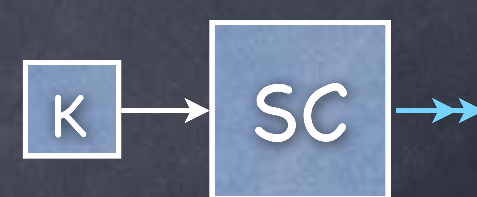

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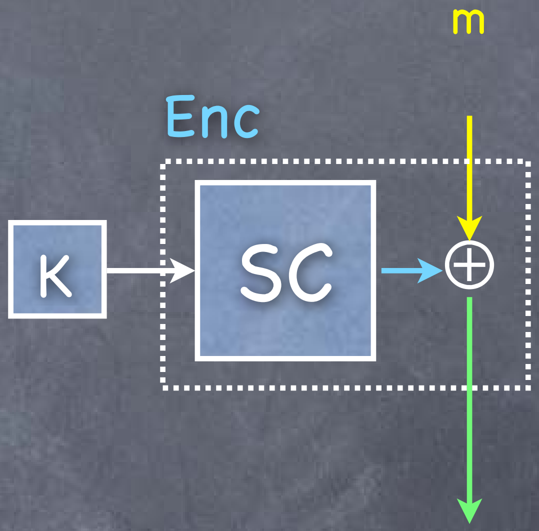

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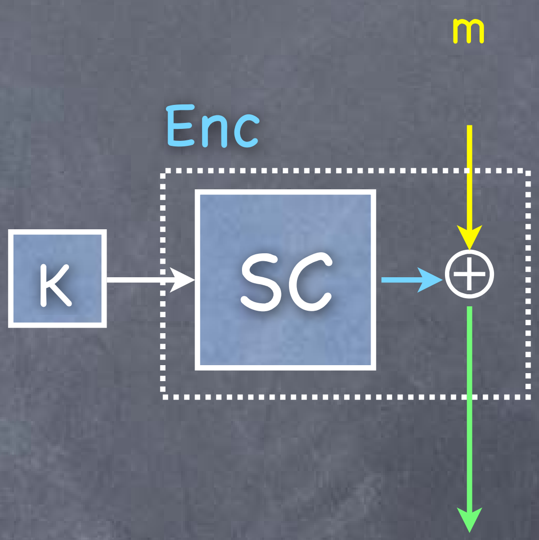

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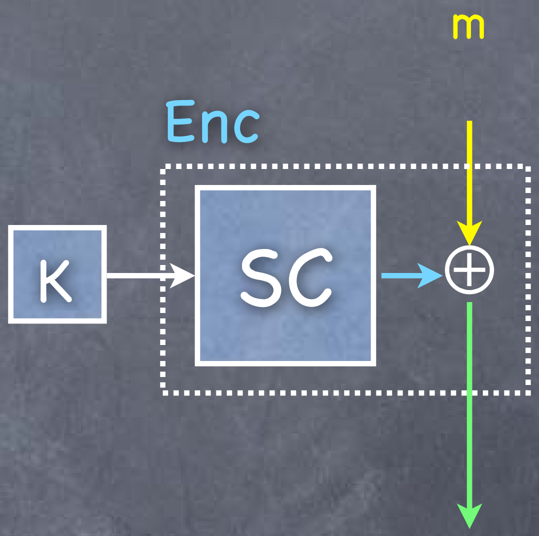

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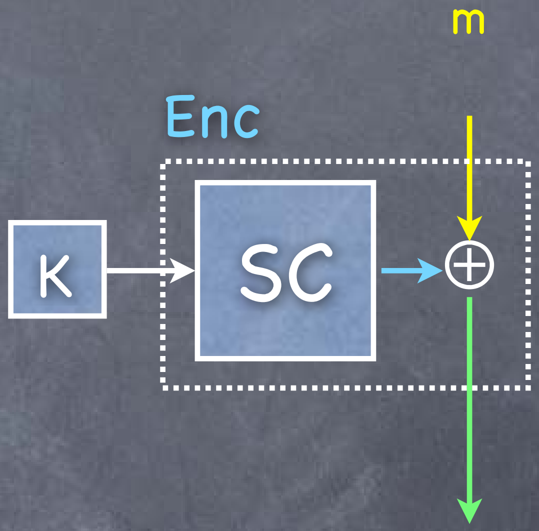

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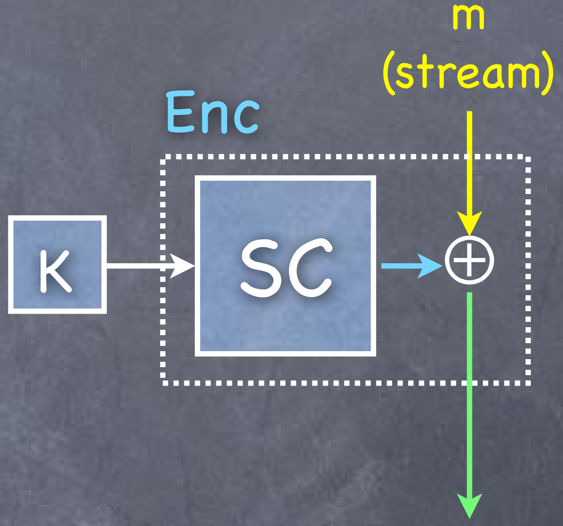

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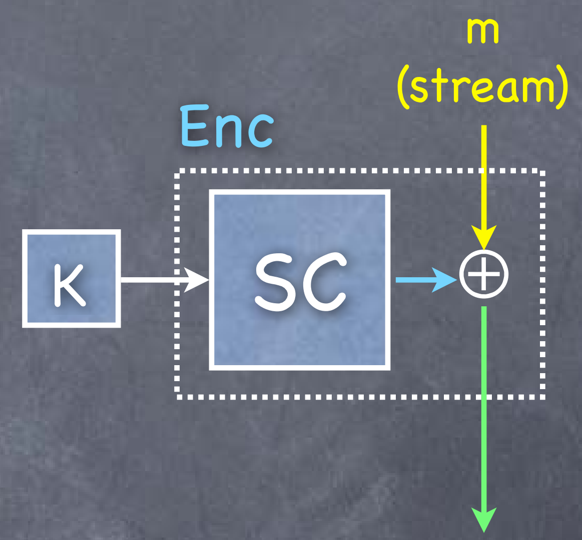

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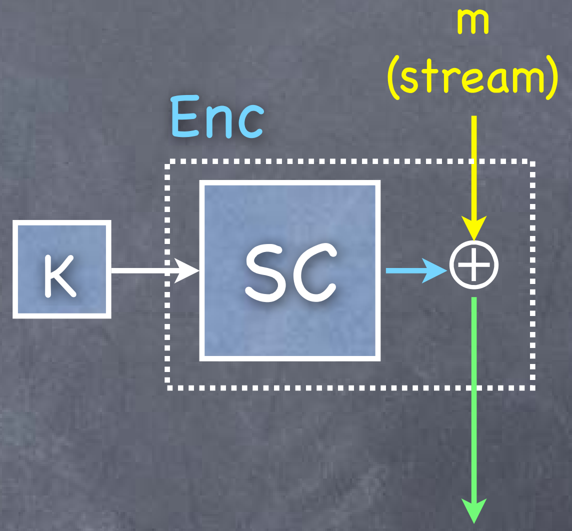

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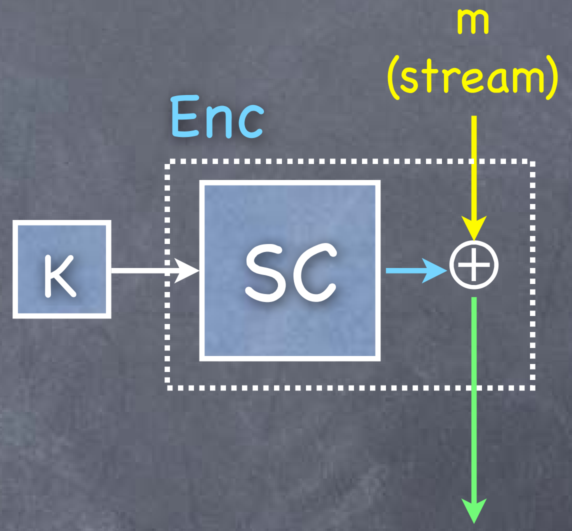

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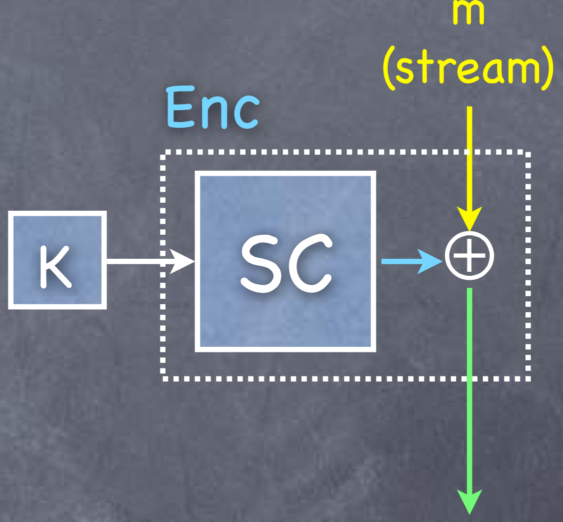

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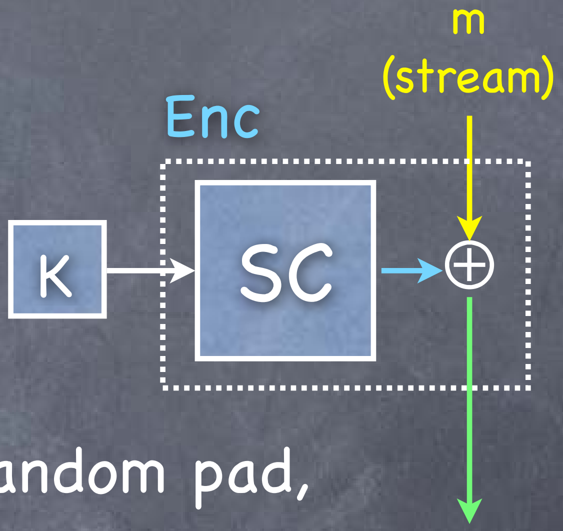

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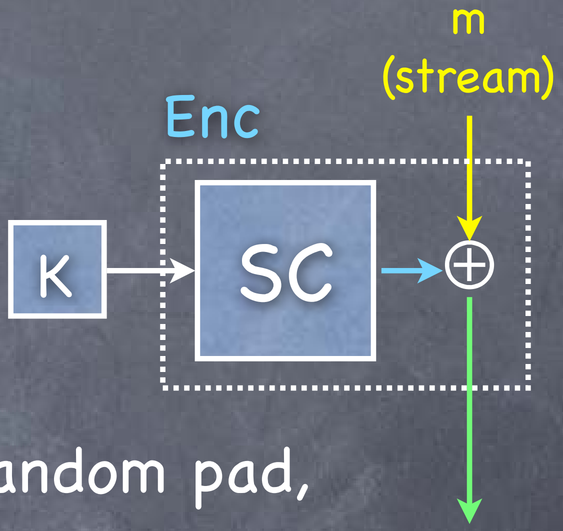

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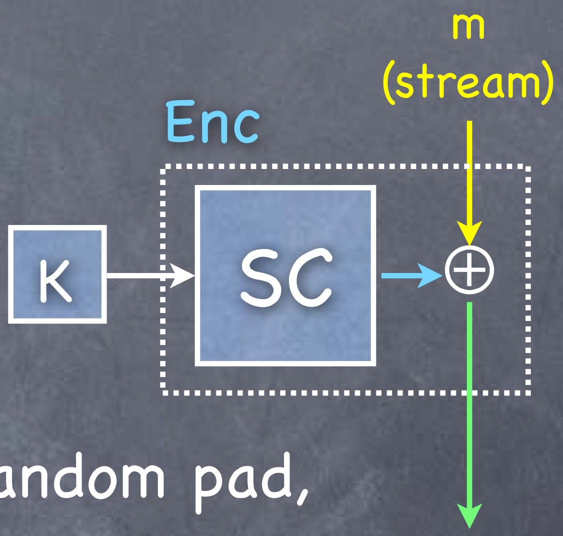

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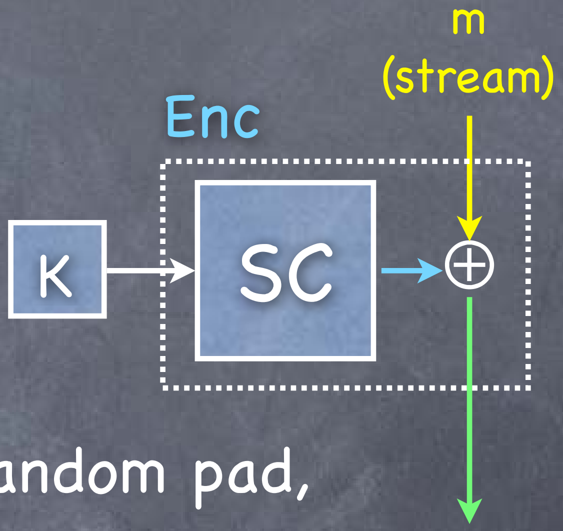
- To show REAL ≈ IDEAL
- Consider an intermediate world, HYBRID:
  - Like REAL, but Enc/Dec use a (long) truly random pad, instead of the output from the stream-cipher
  - HYBRID = IDEAL (recall perfect security of one-time pad)
  - Claim: REAL ≈ HYBRID

One-time CPA-secure SKE with a Stream-Cipher



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    - Consider the experiments as a system that accepts the pad from outside (R' = SC(K) for a random K, or truly random R) and outputs the environment's output. This system is PPT, and so can't distinguish pseudorandom from random.