Homomorphic Encryption

Lecture 15

And some applications

Homomorphic Encryption

Homomorphic Encryption

- Group Homomorphism: Two groups G and G′ are homomorphic if there exists a function (homomorphism) $f: G \rightarrow G'$ such that for all $x, y \in G$, $f(x +_G y) = f(x) +_{G'} f(y)$

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- Not covered: Fully Homomorphic Encryption, which supports ring homomorphism (addition <u>and</u> multiplication of messages)

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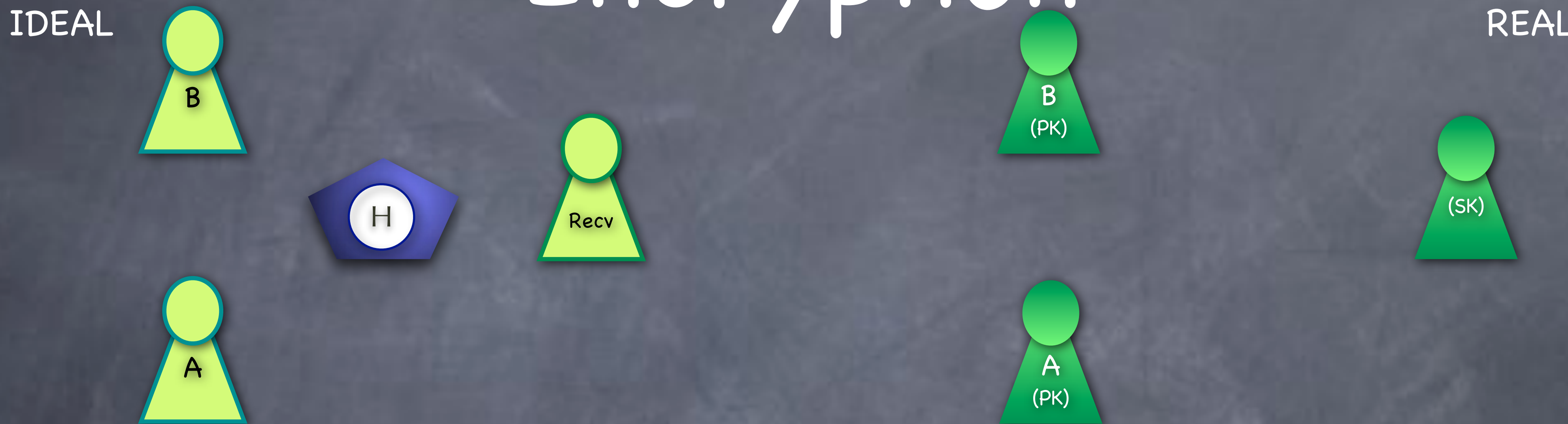
  - Rerandomization useful even without homomorphism

Unlinkable Homomorphic Encryption



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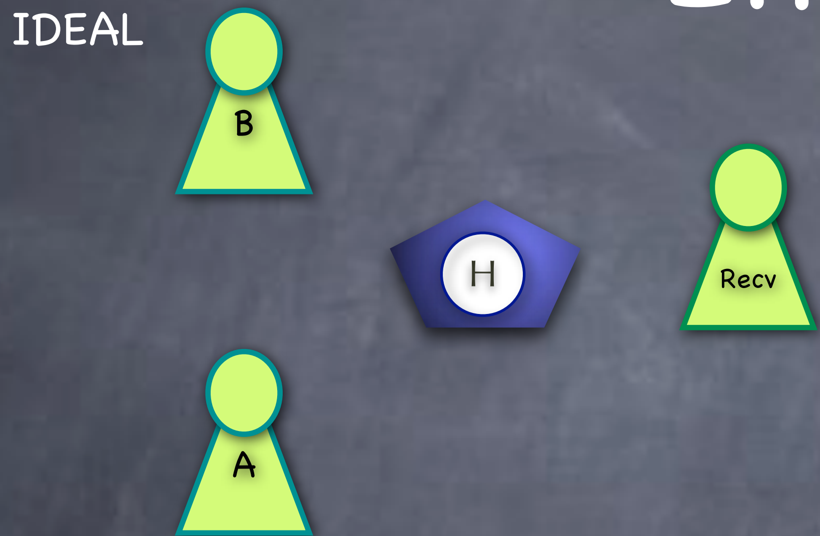

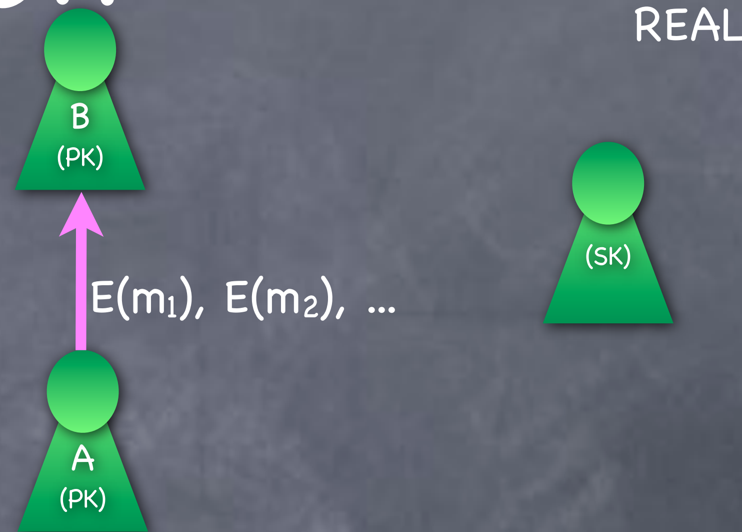

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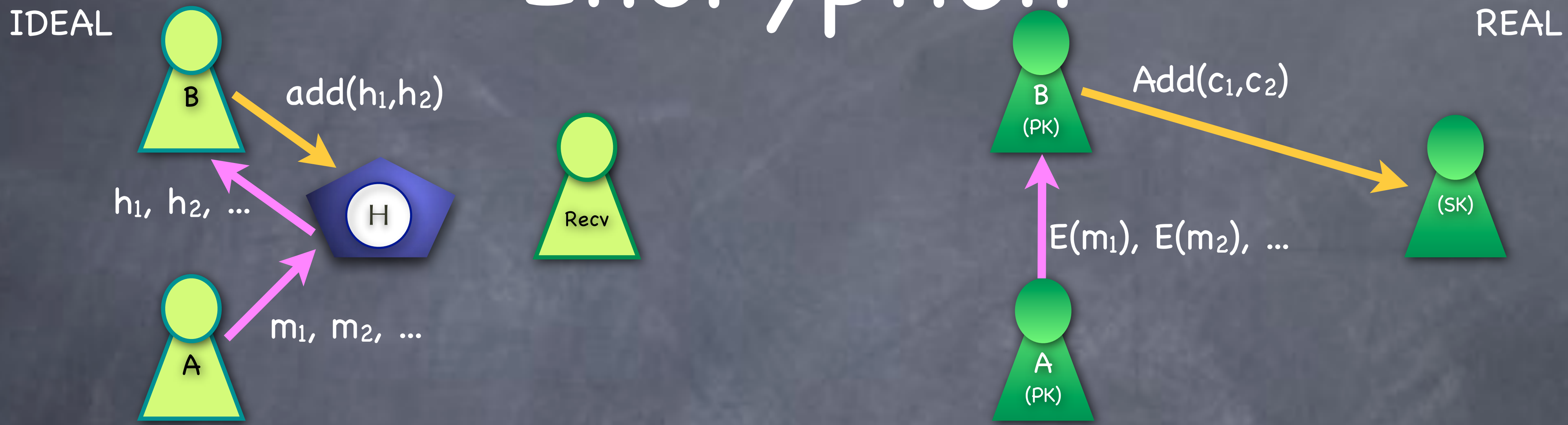

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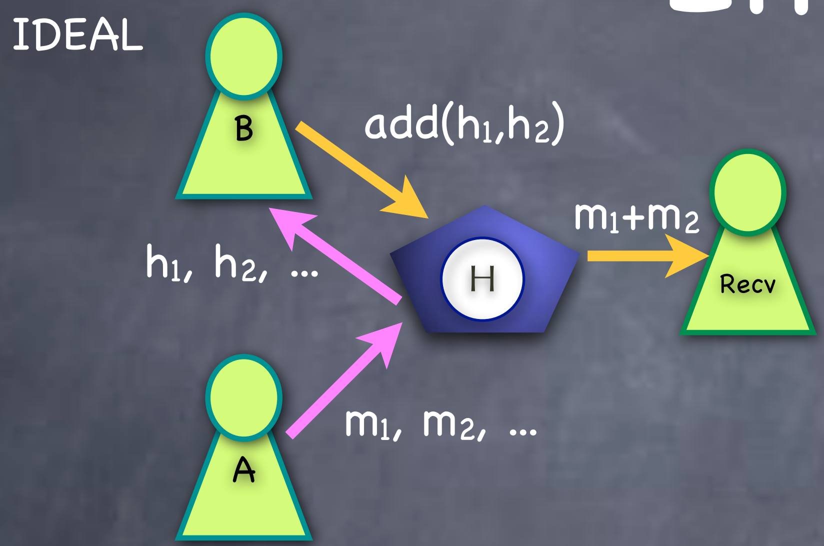

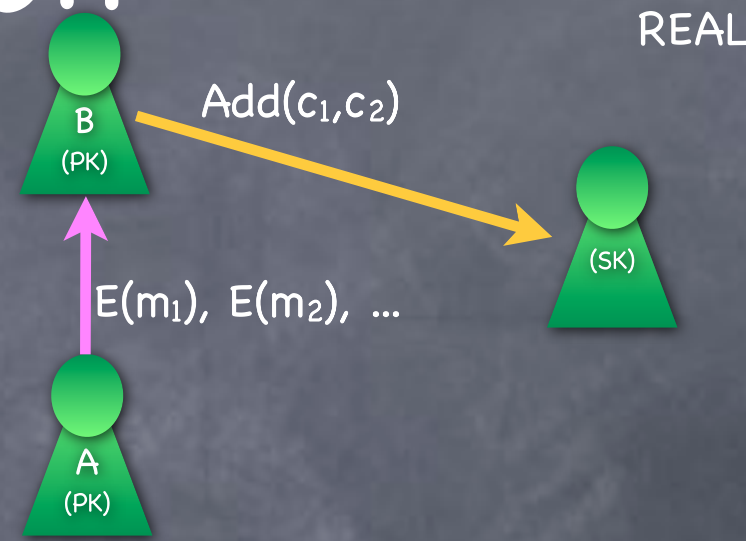

- Considers only passive corruption
- Functionality gives "handles" to messages posted; accepts requests for posting fresh messages, or derived messages
- Unlinkability: Above, (honest-but-curious) receiver gets only the message m_1+m_2 in IDEAL; is not told if it is a fresh message or derived from other messages

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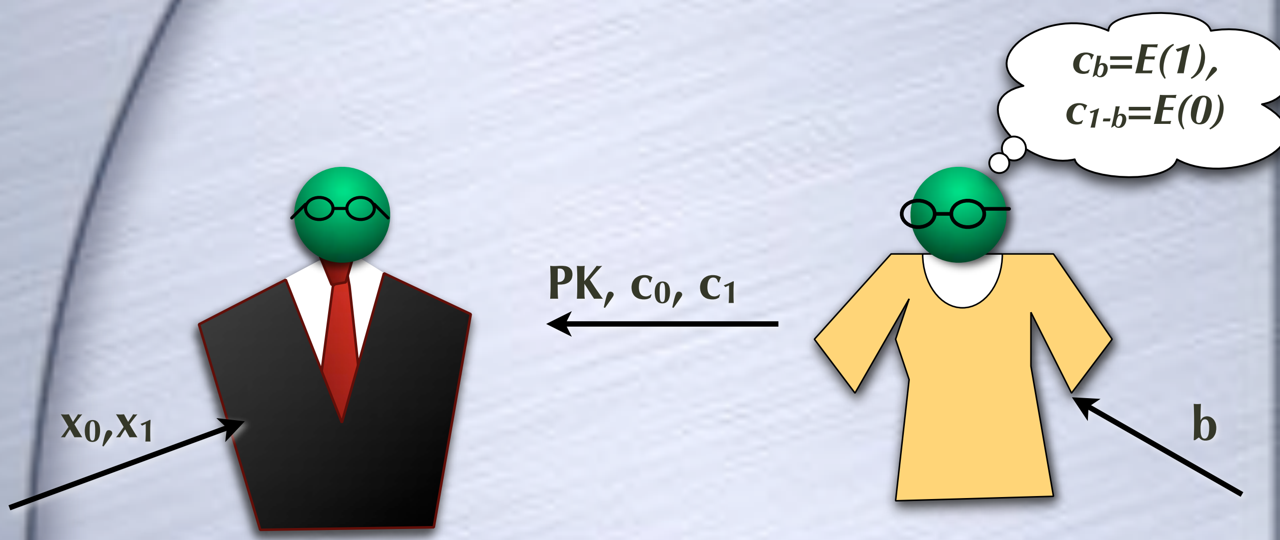

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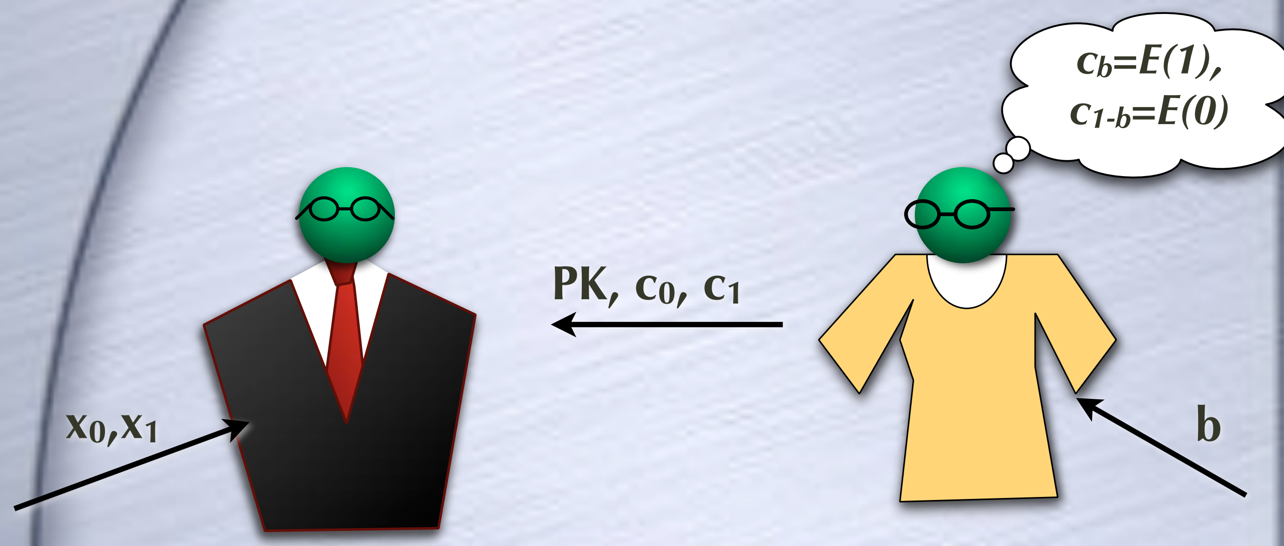

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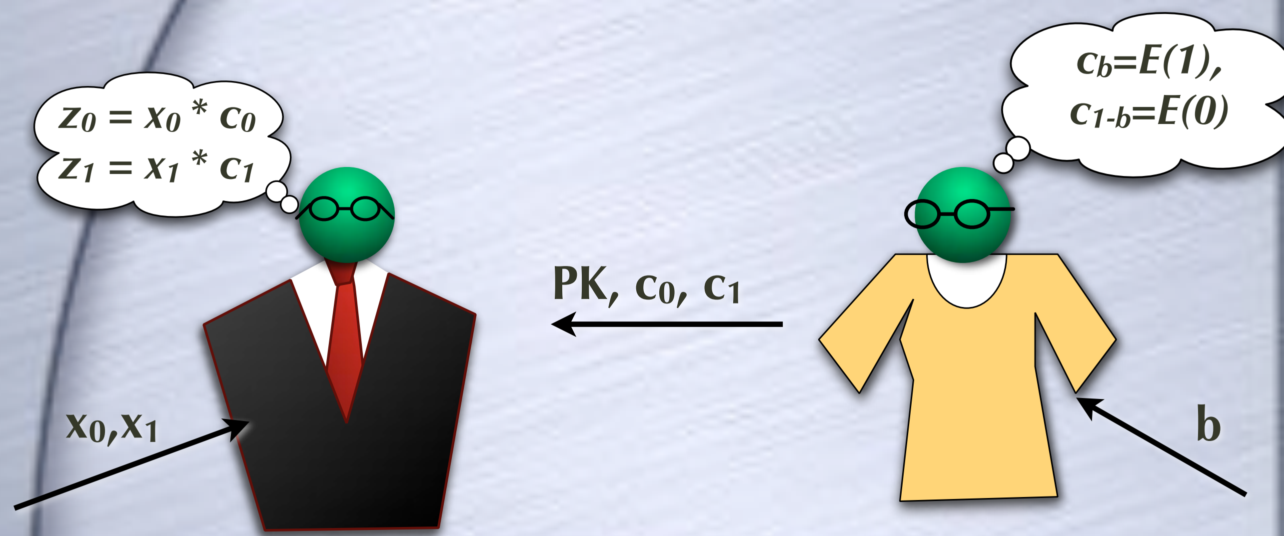

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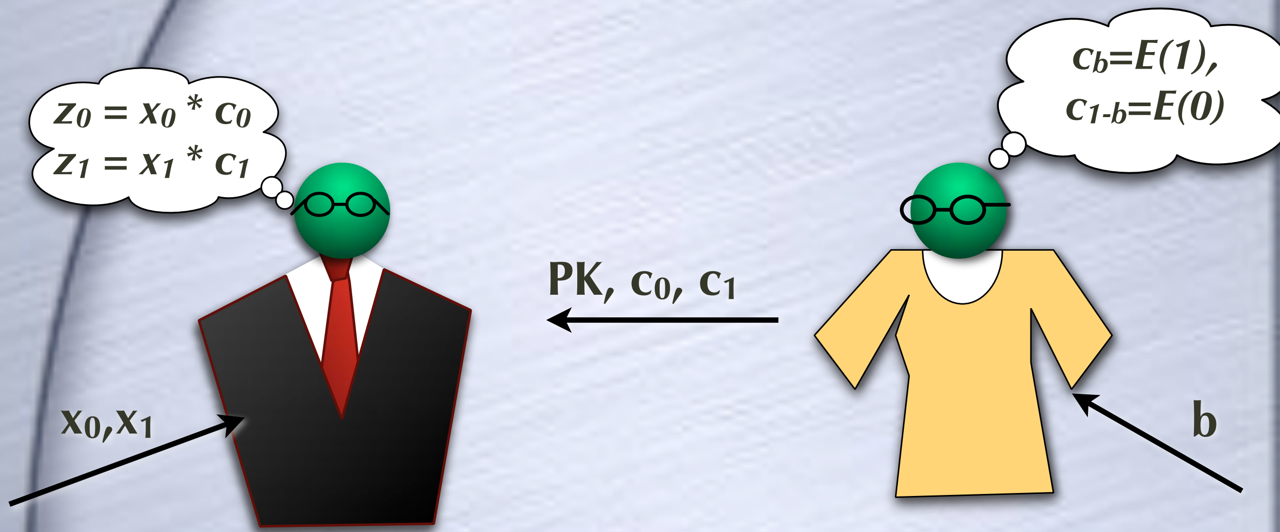

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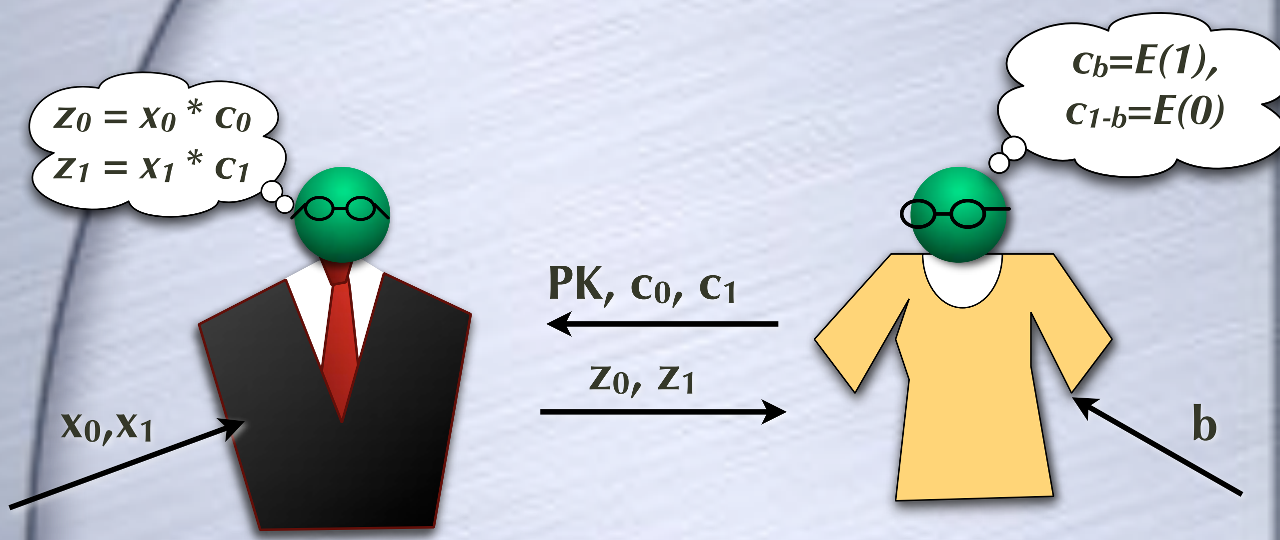

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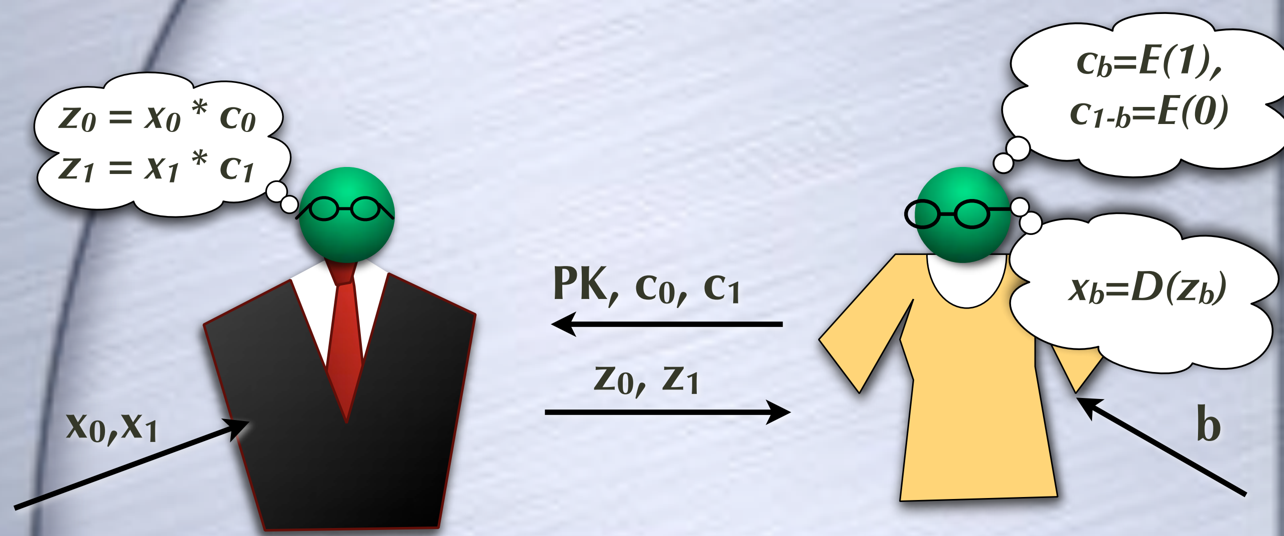

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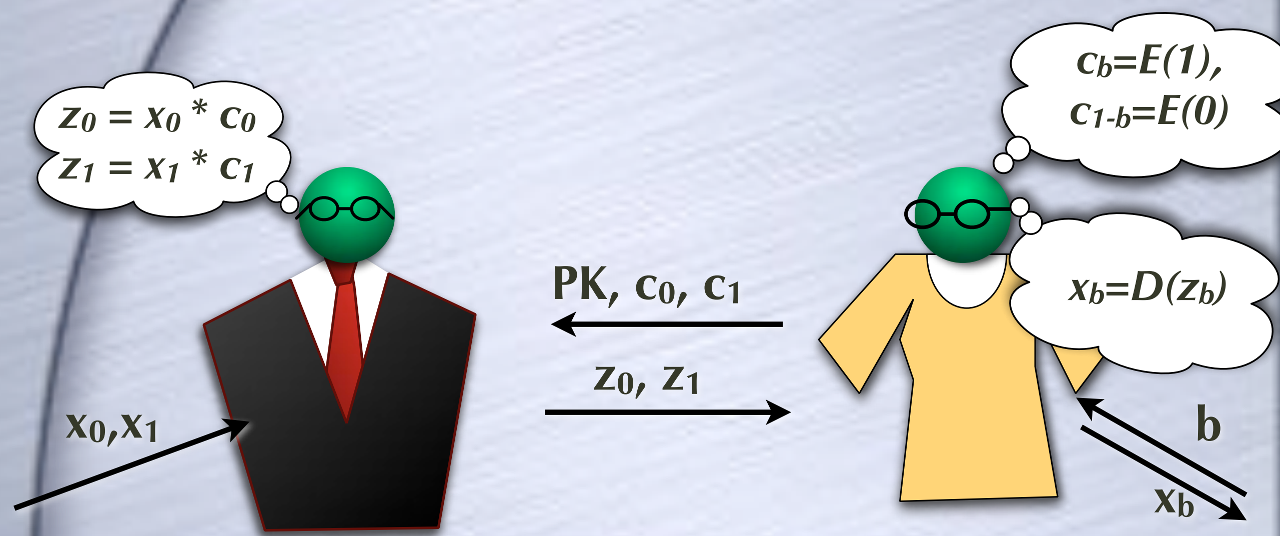

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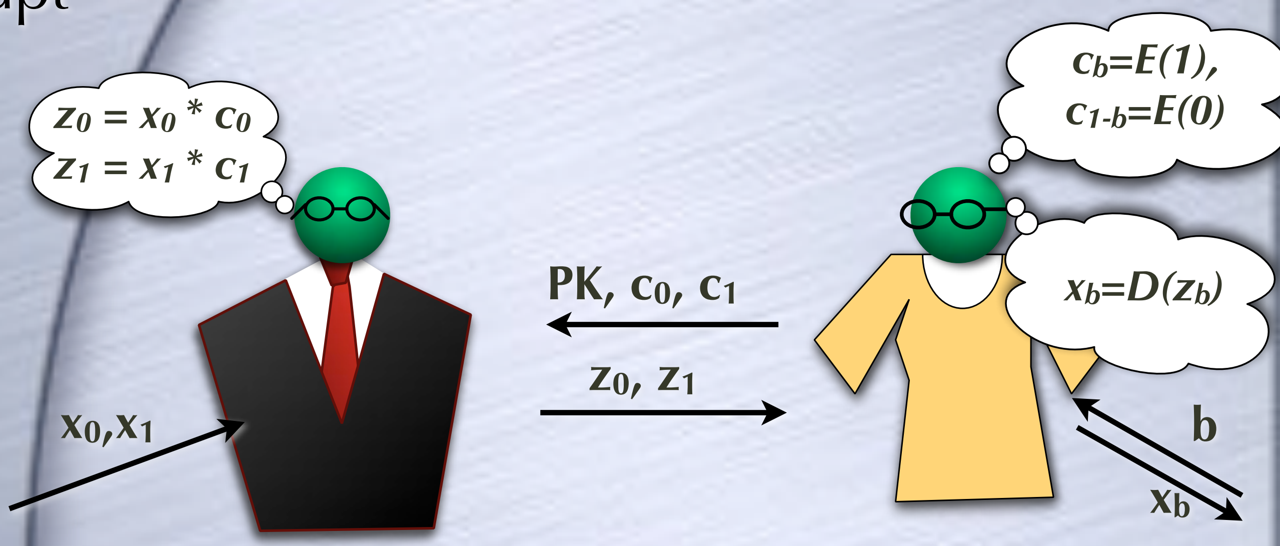

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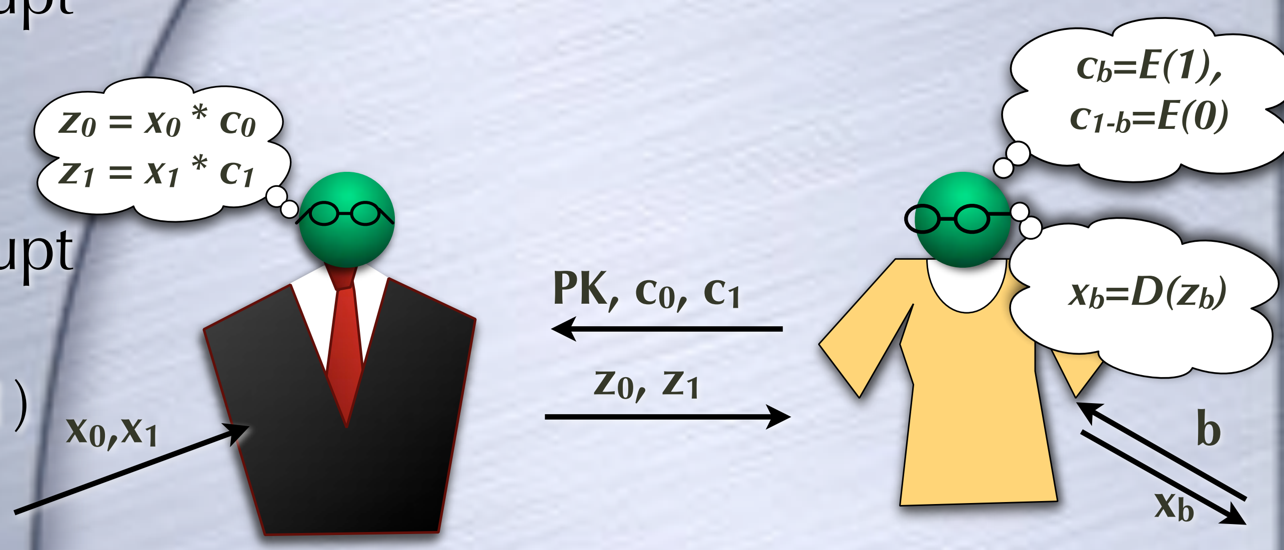
- Simulation for passive-corrupt sender: Extract x_0,x_1 by setting both c_0 and c_1 to E(1)



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- Tool: Additively homomorphic encryption

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- Unlinkability: ReRand(c) = c.Enc(0)

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  - In the following: database values are integers in [0,m); homom. enc. over a group with an element 1 s.t. $ord(1) \geq m$. For integer x and ciphertext $\underline{c}$, define x*c using "repeated doubling": $0*\underline{c} = E(0)$; $1*\underline{c} = \underline{c}$; $(a+b)*\underline{c} = Add(a*\underline{c}, b*\underline{c})$.

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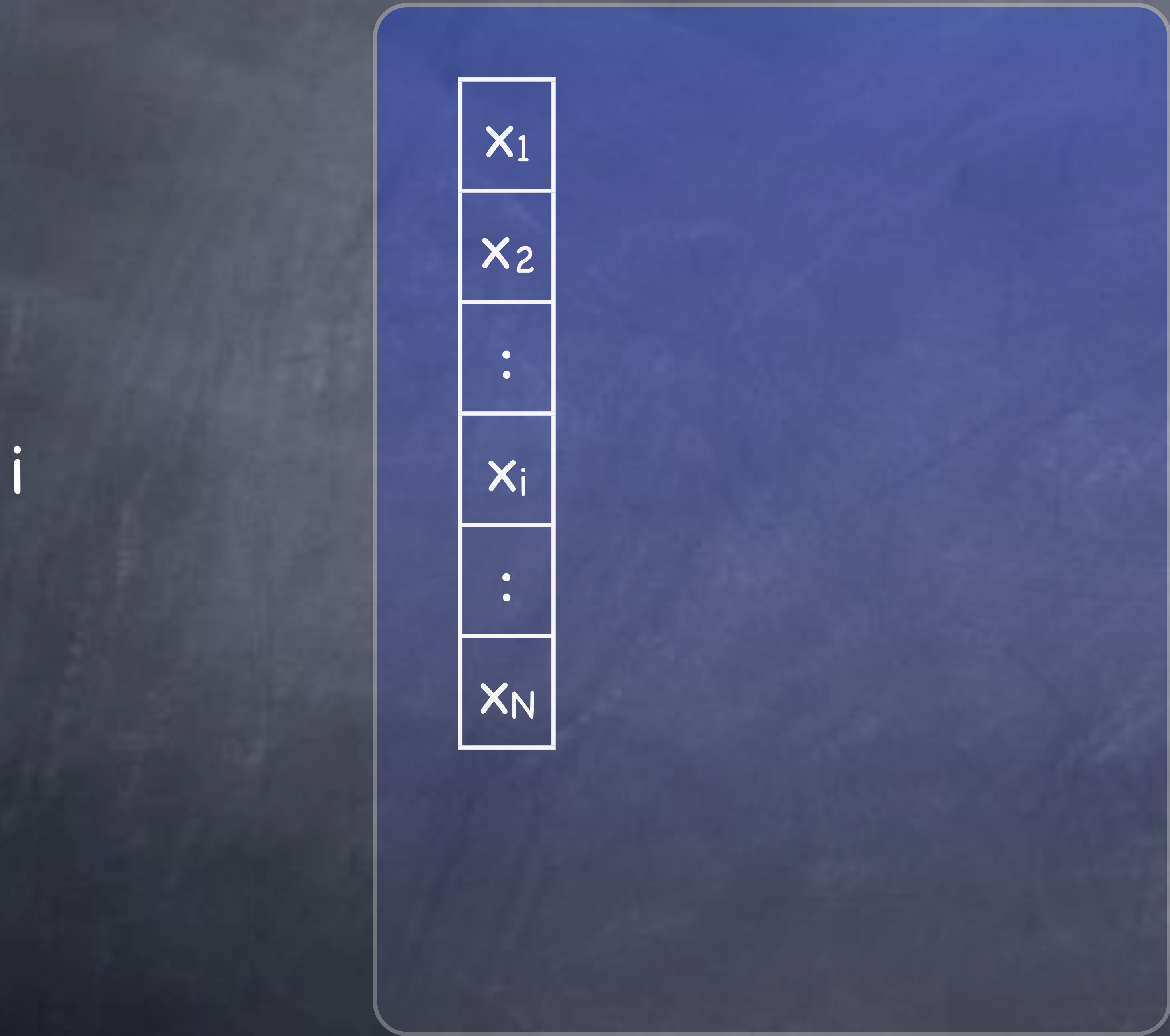

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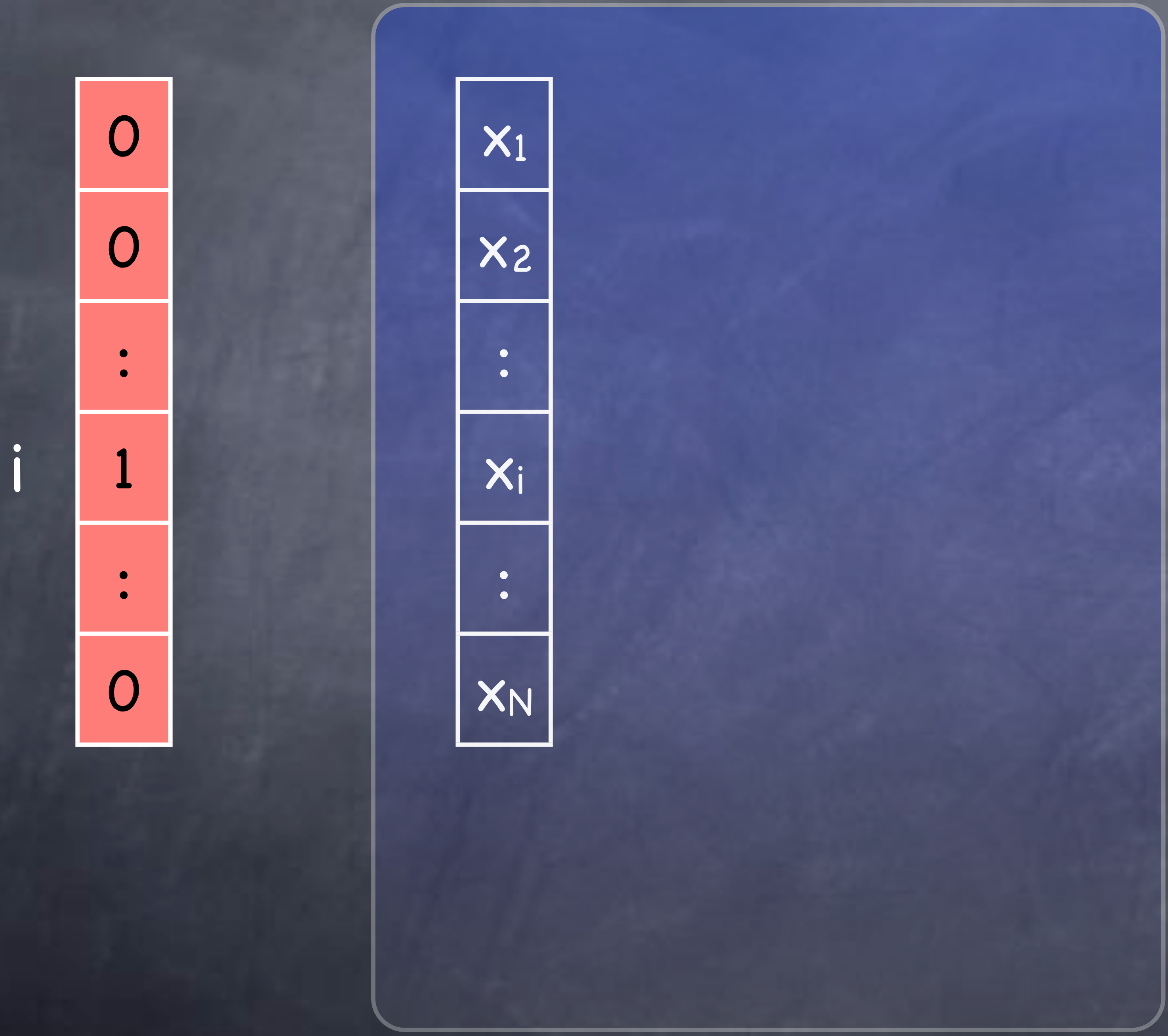

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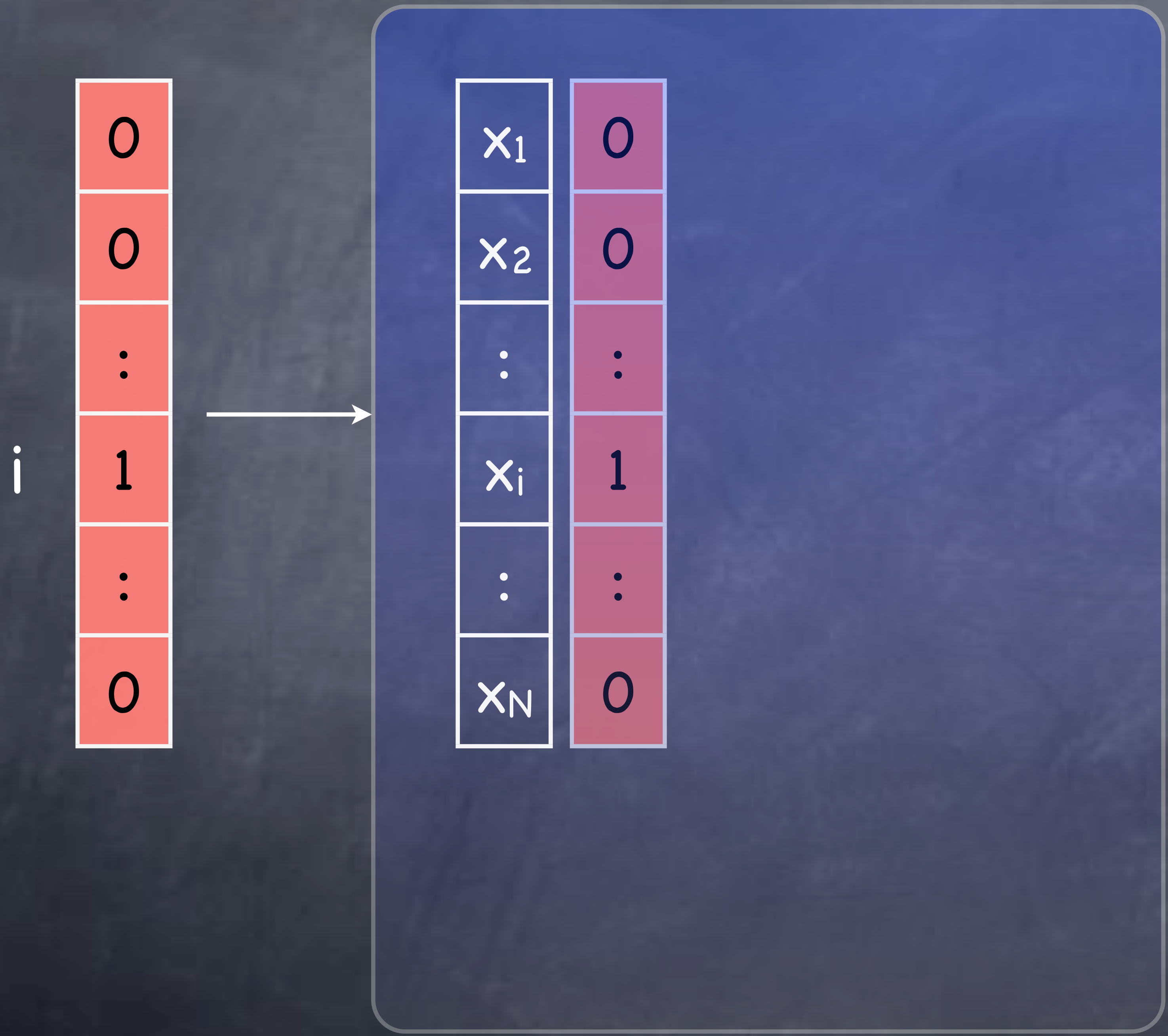

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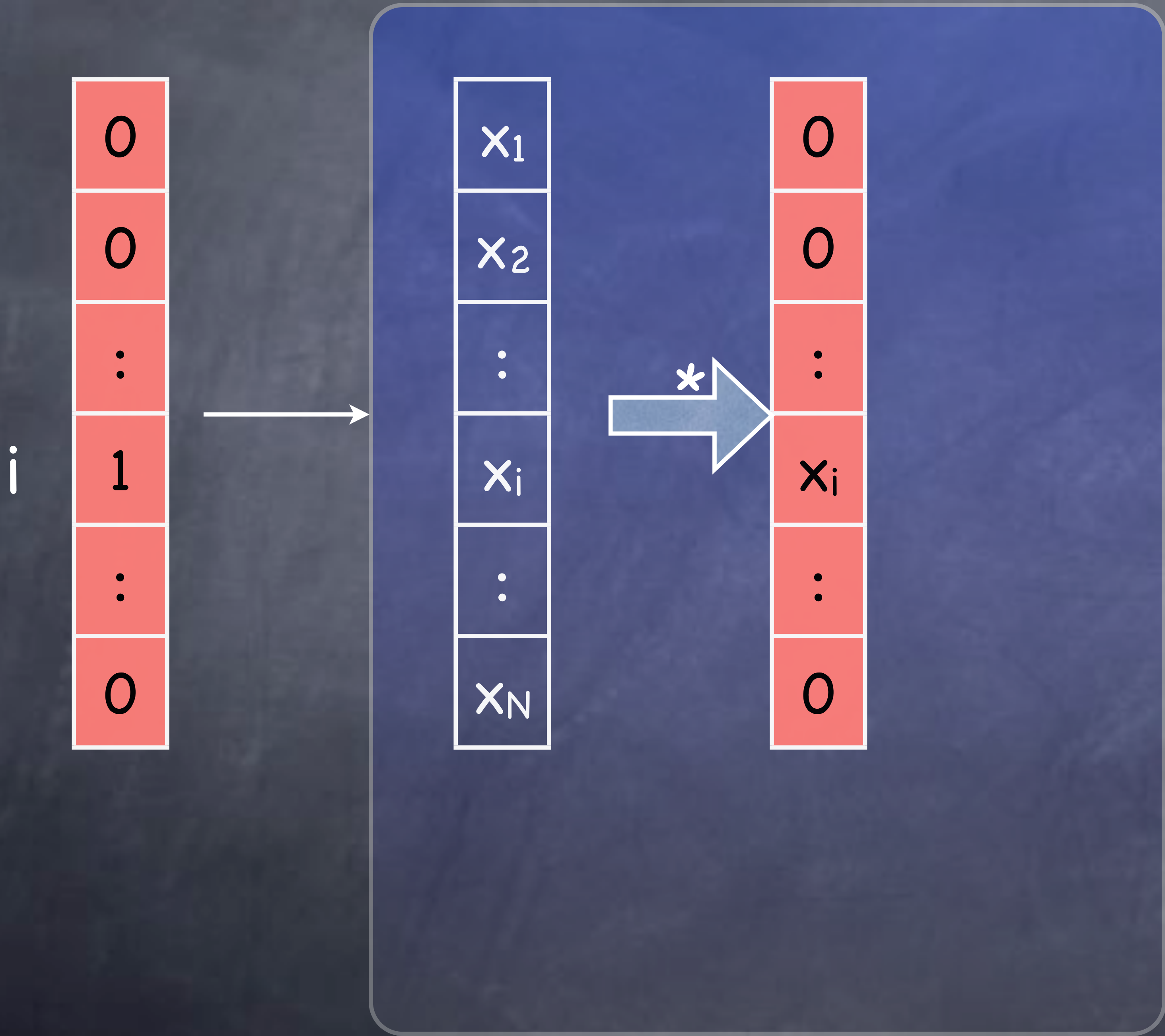

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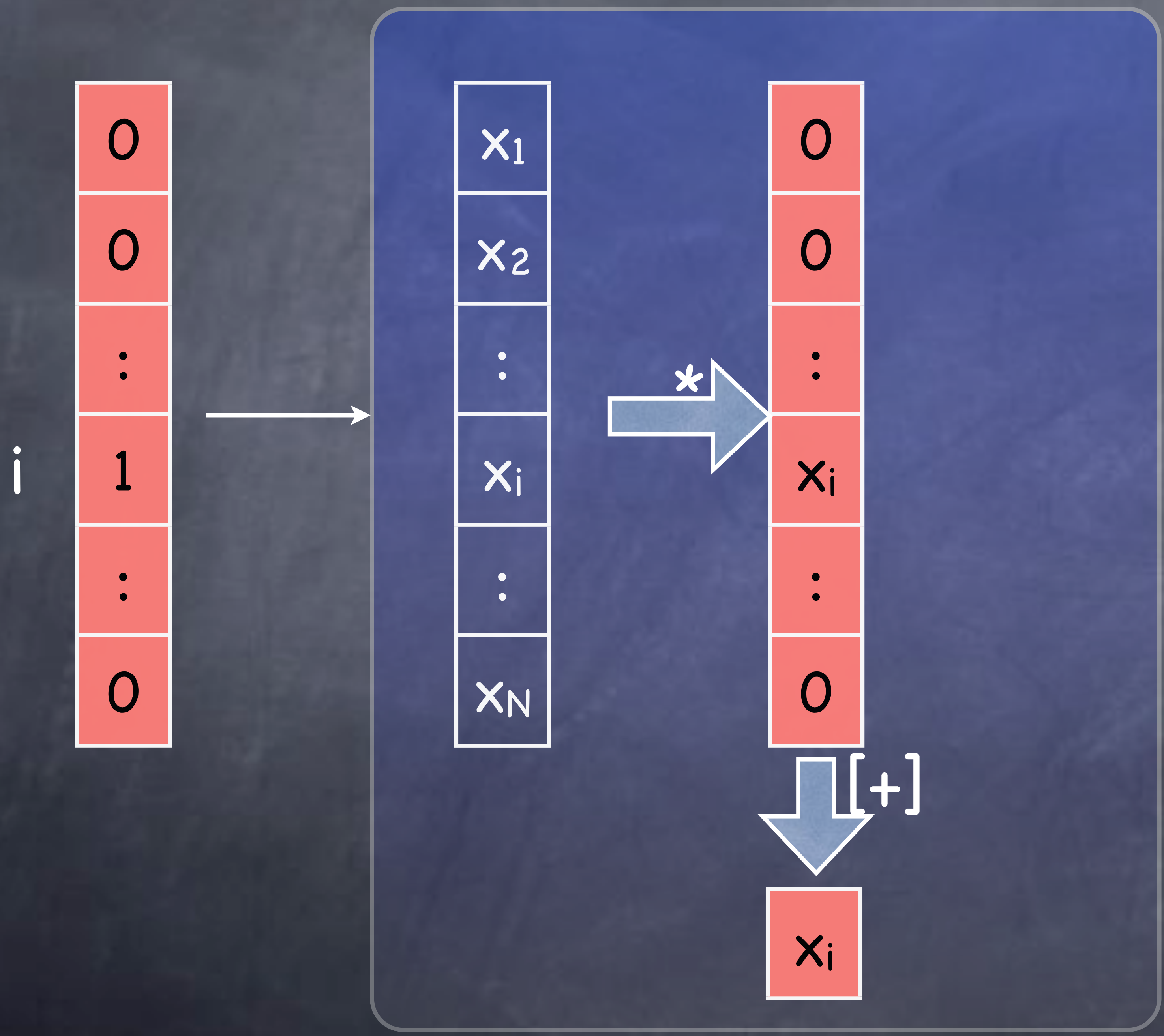

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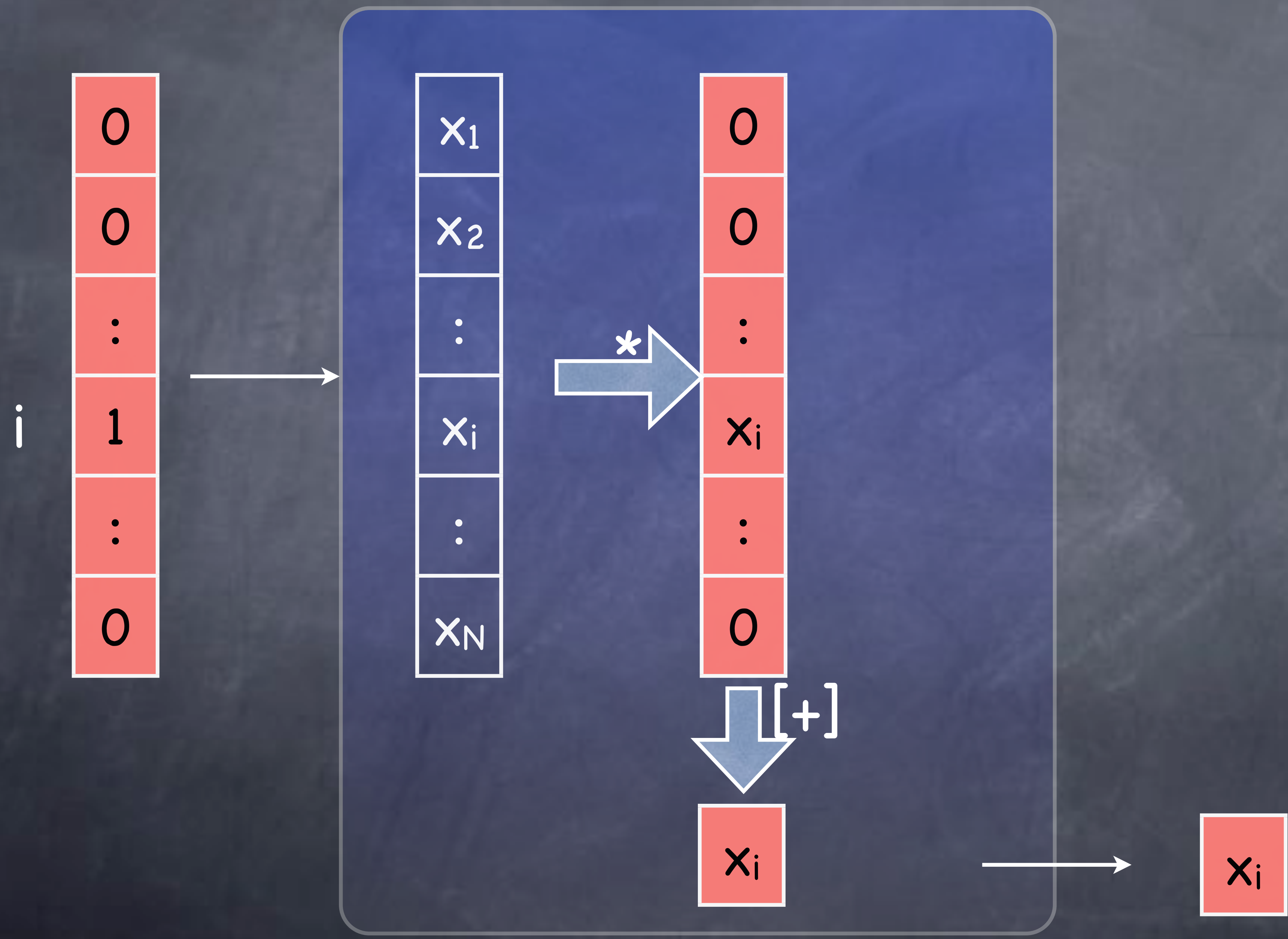

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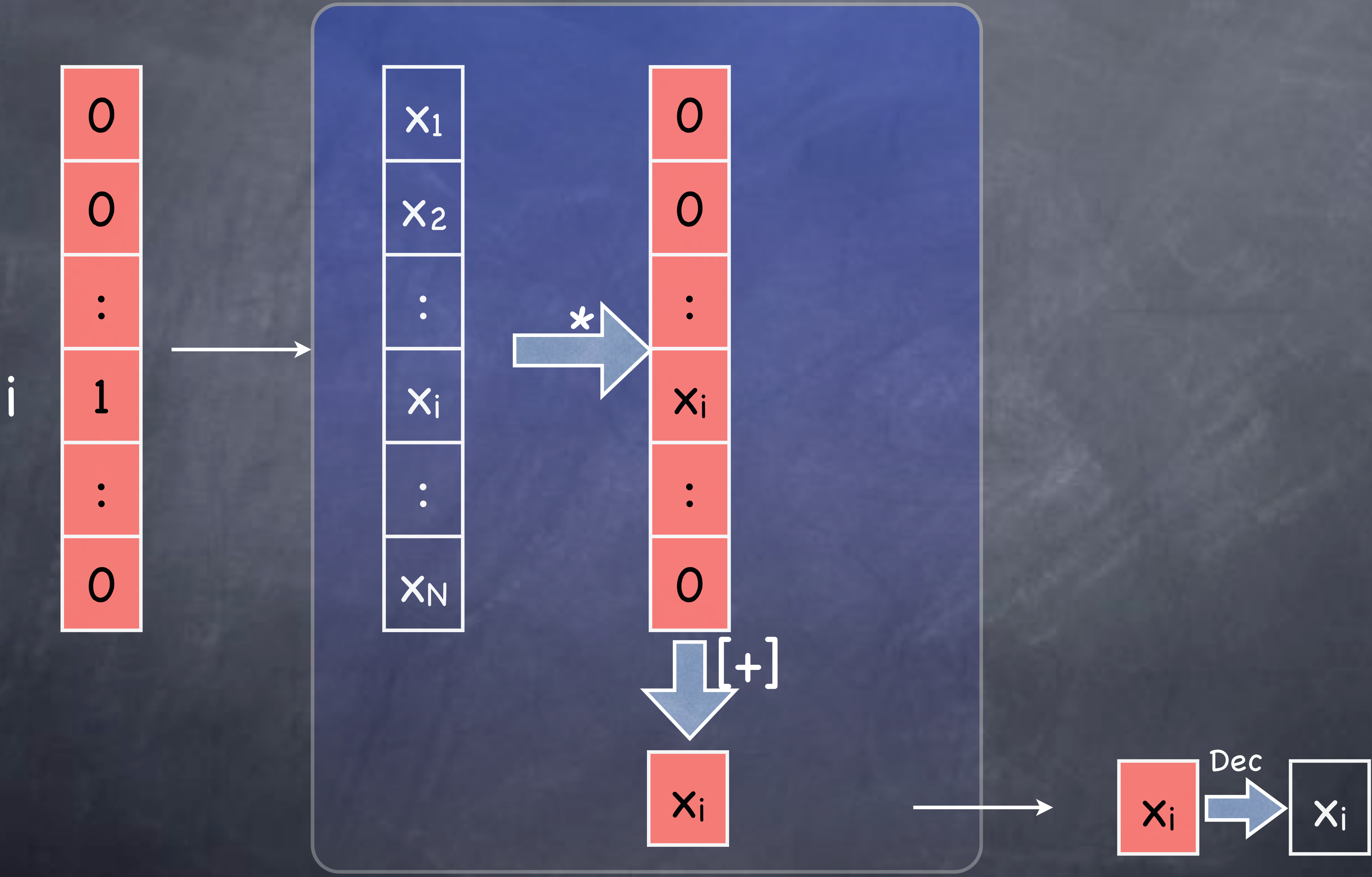

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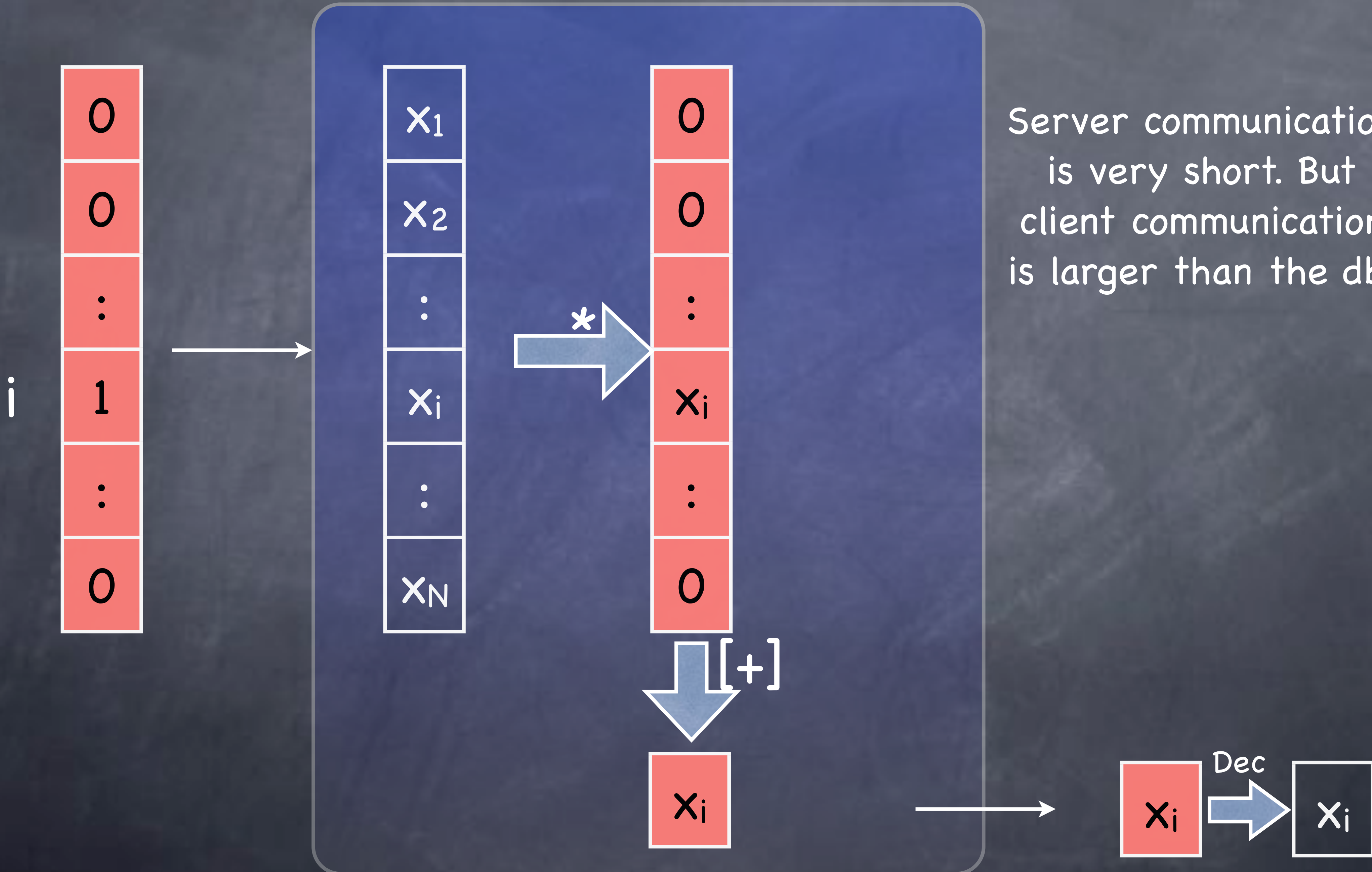

Server communication is very short. But client communication is larger than the db!

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x_{11}					x_{1N}
x_{21}					x_{2N}
:					:
x_{i1}			x_{ij}		x_{iN}
:					:
x_{N}					x_{NN}

Private Information Retrieval

0
0
:
1
:
0

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x_{21}					x_{2N}
:					:
x_{i1}			x_{ij}		x_{iN}
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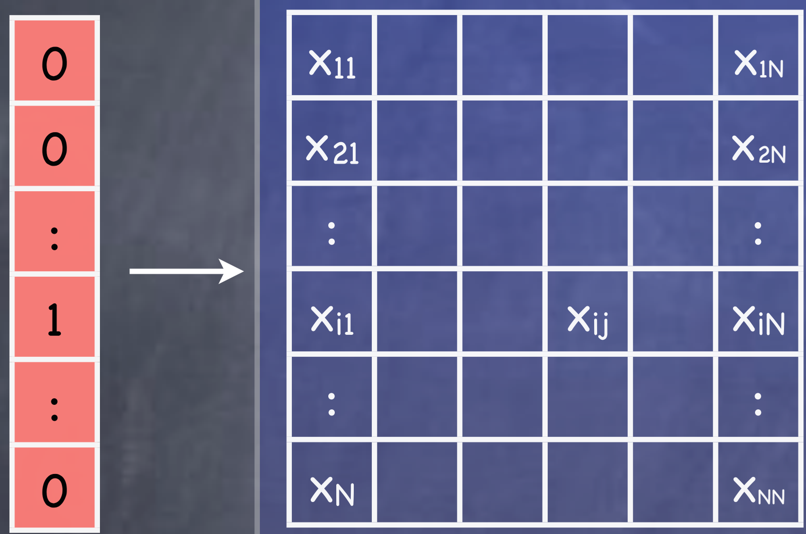

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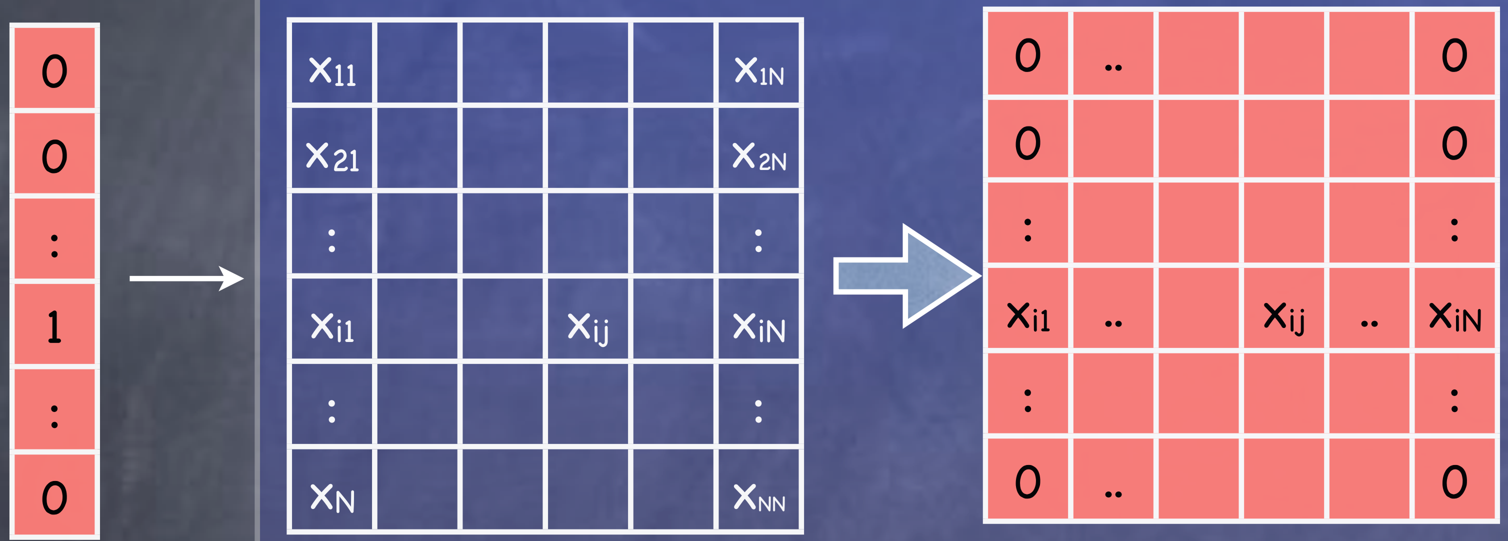

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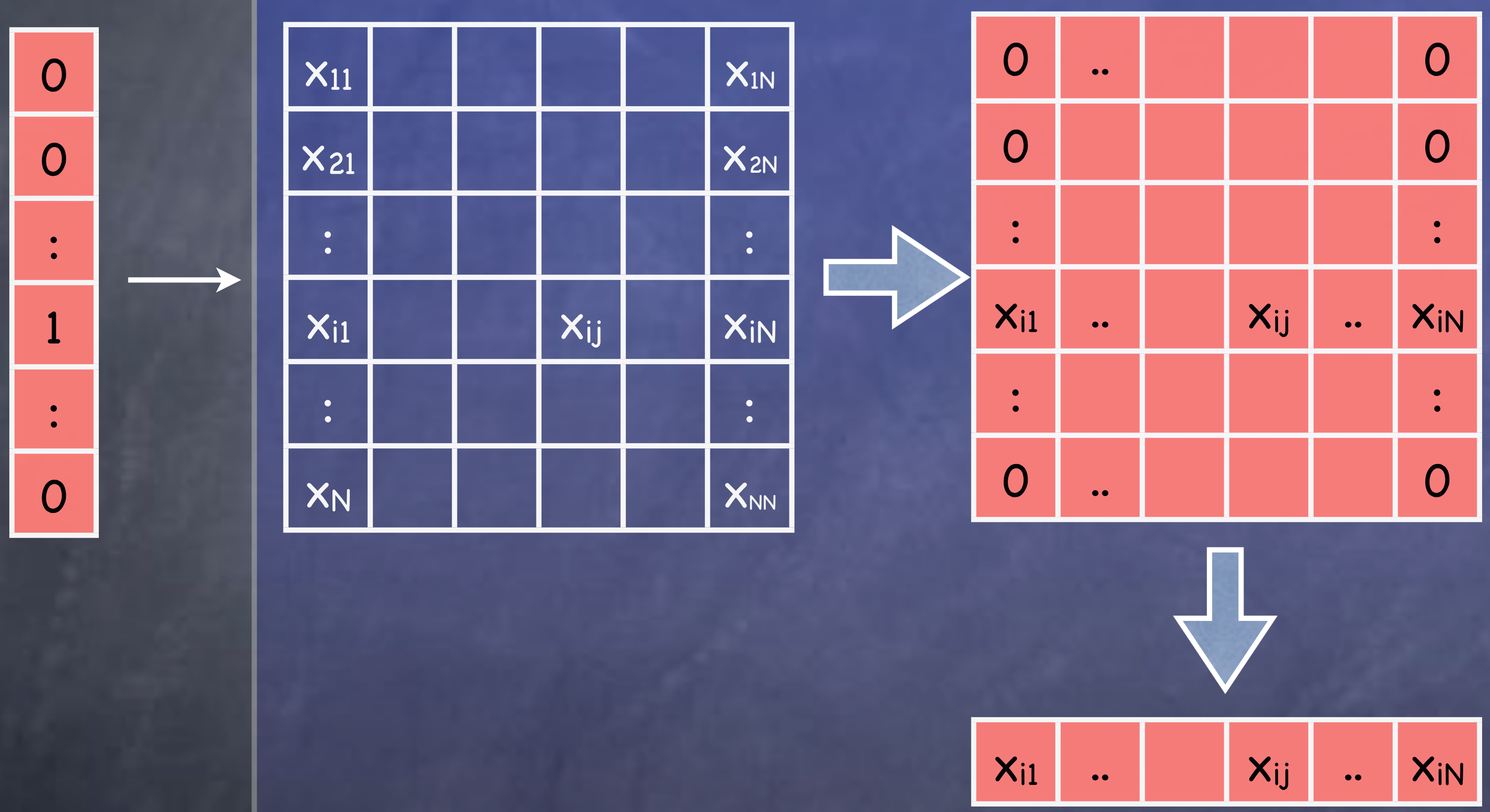

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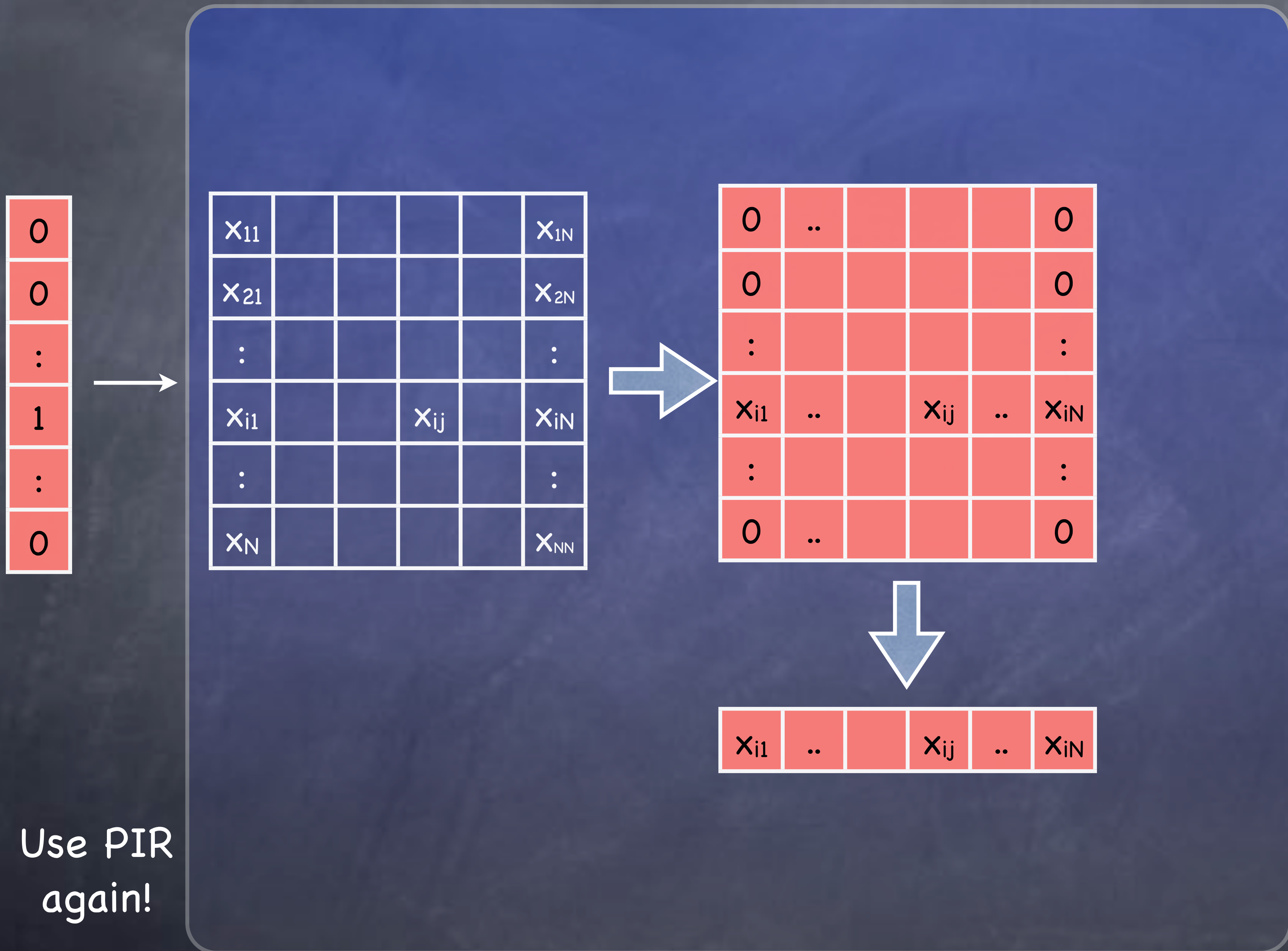

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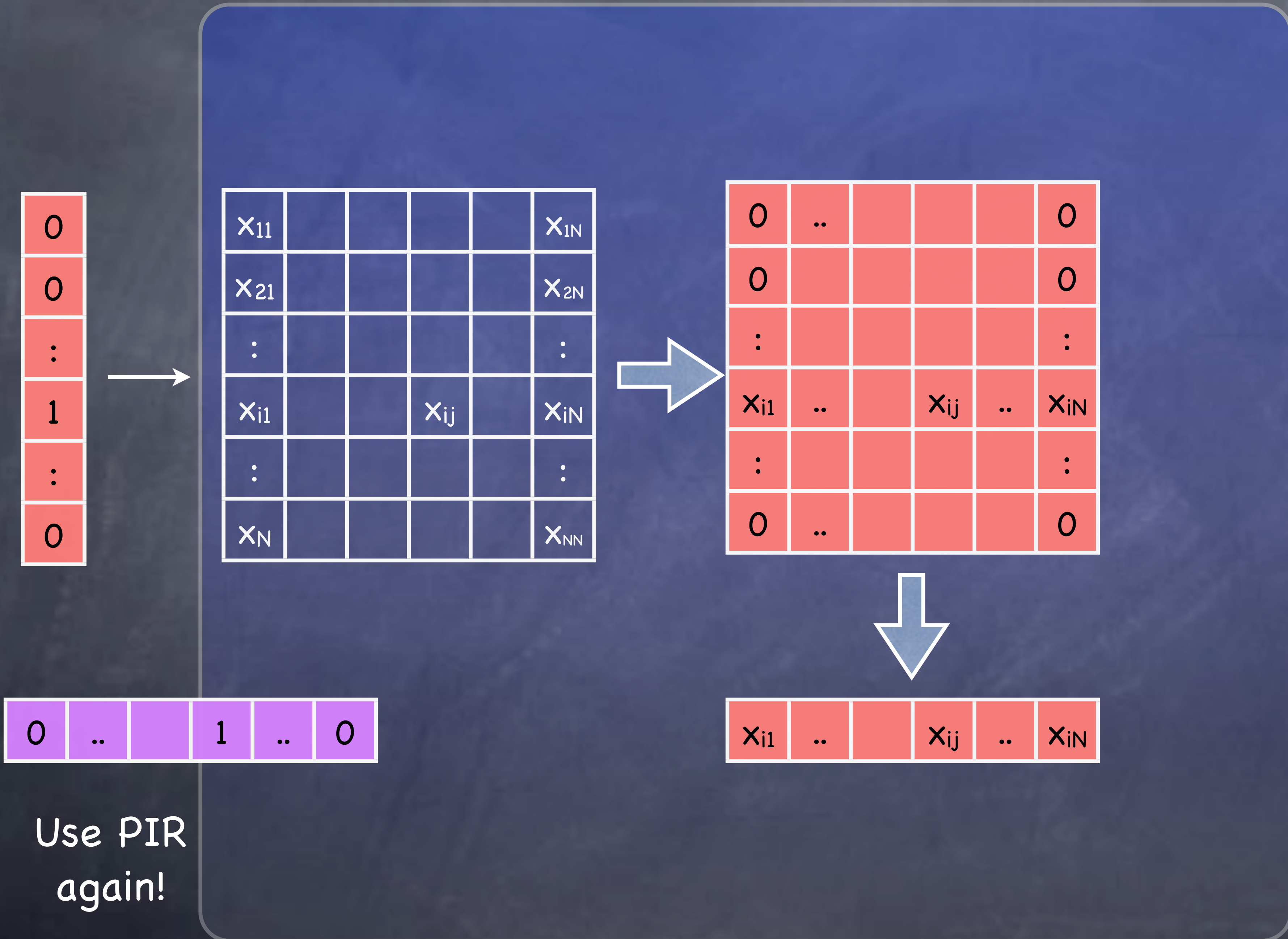

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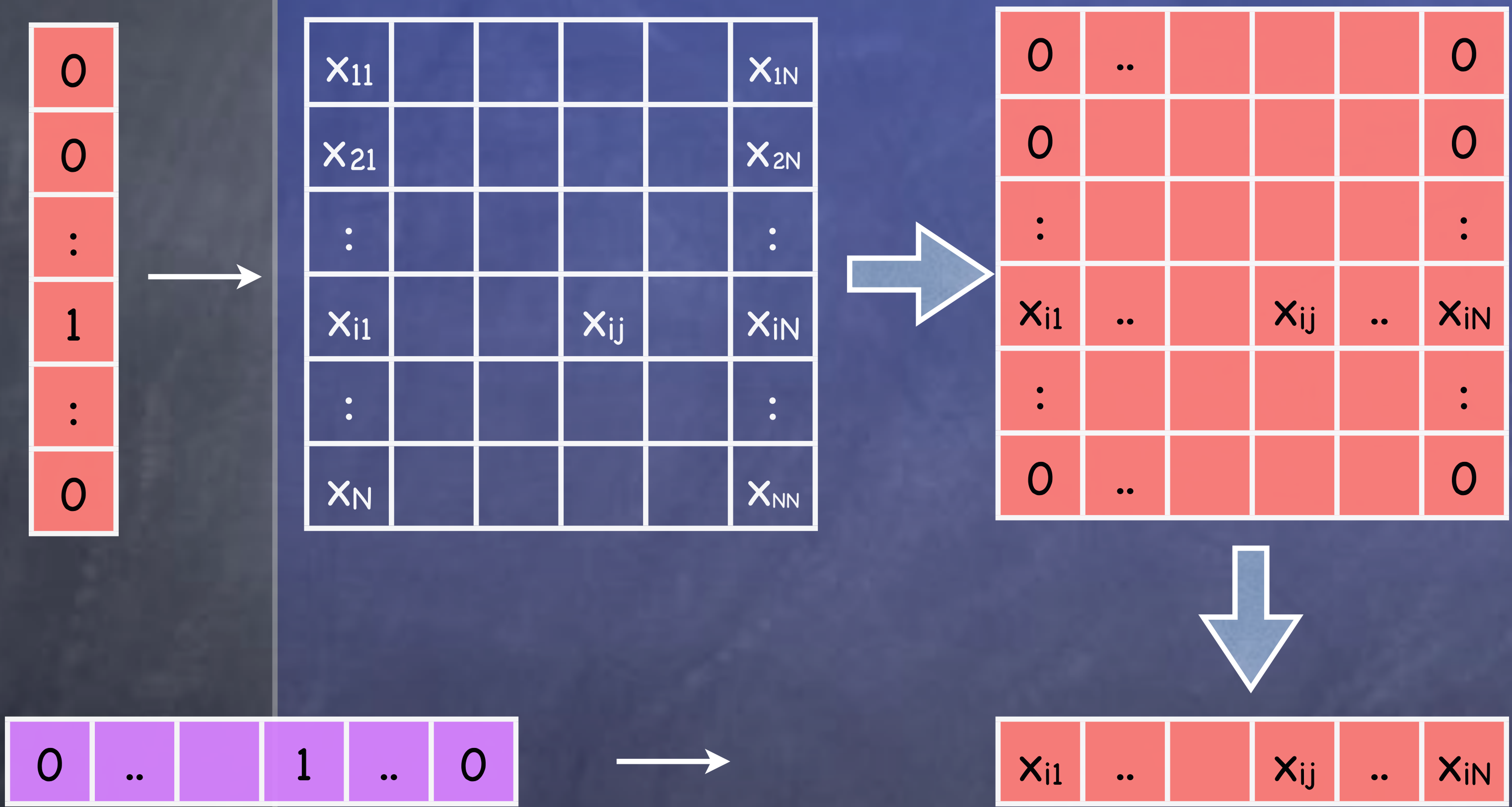

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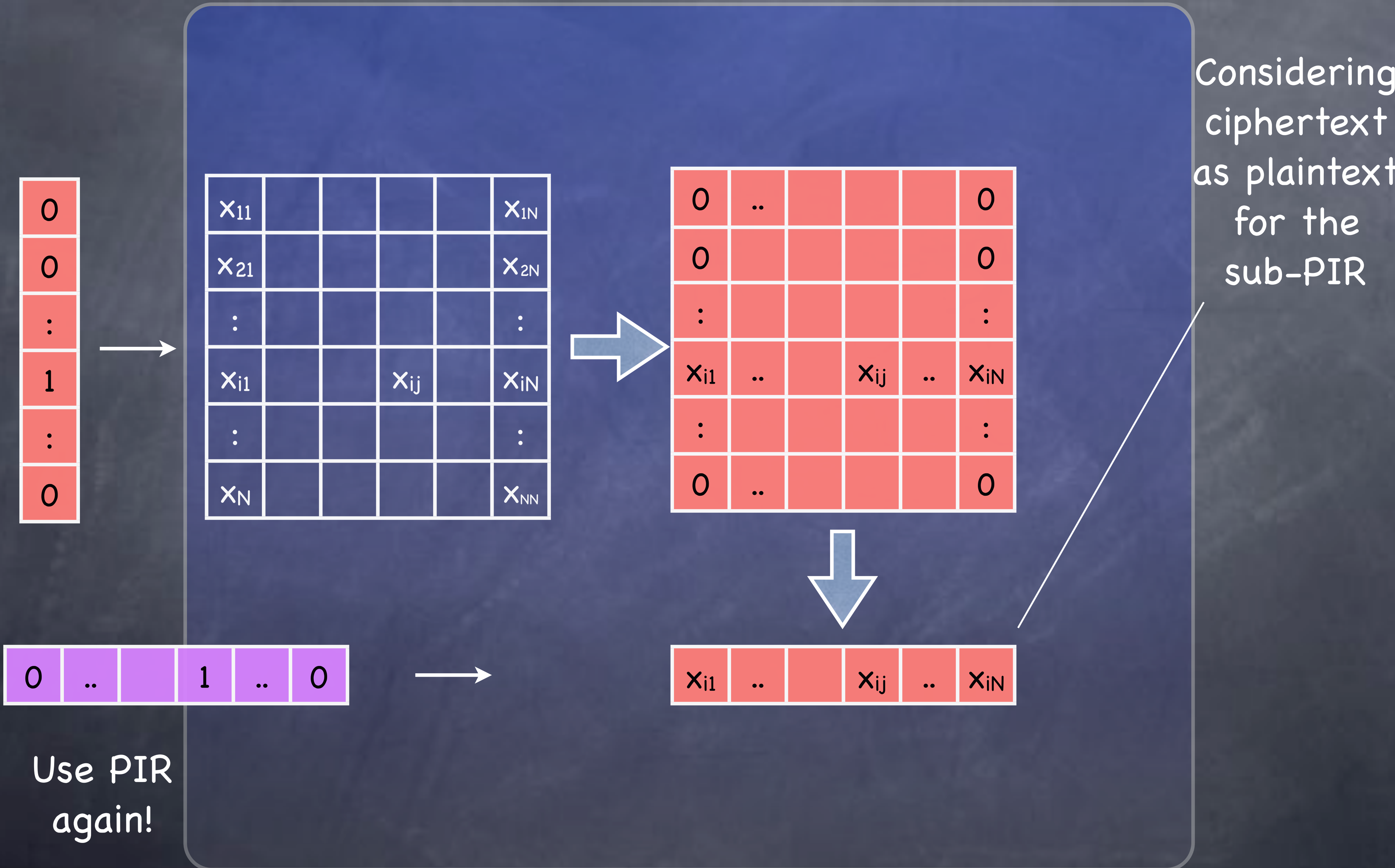

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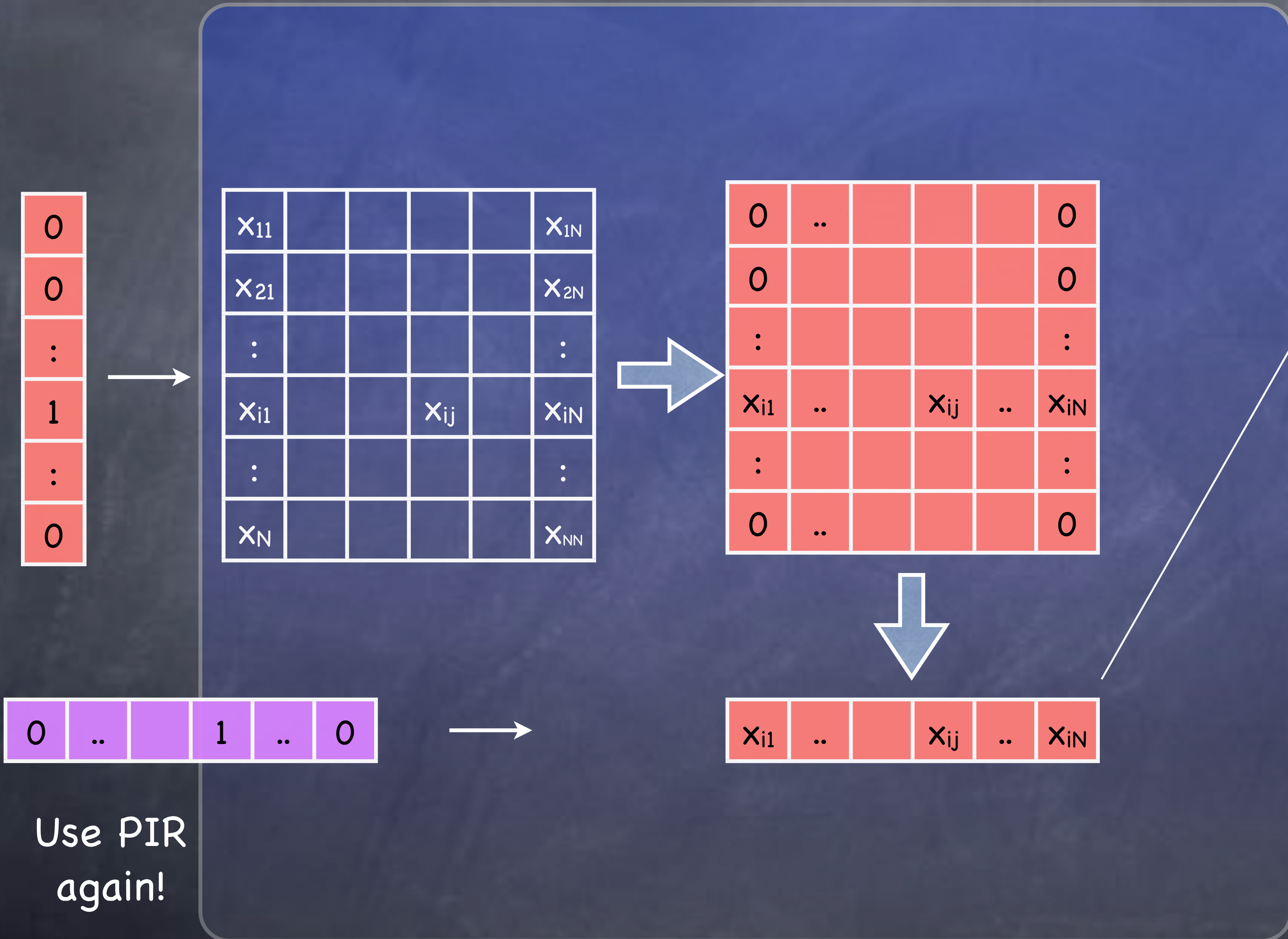

Considering ciphertext as plaintext for the sub-PIR

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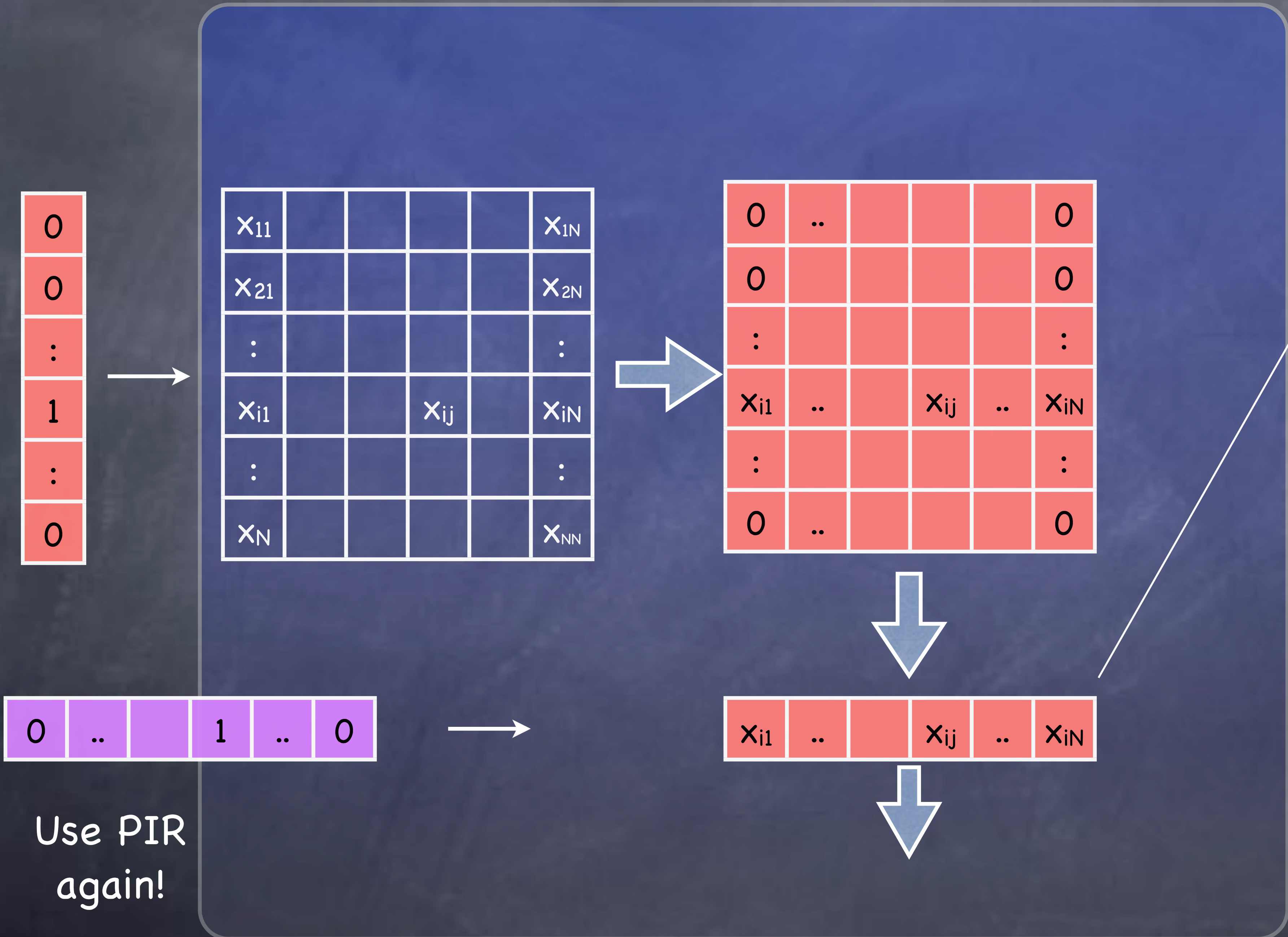

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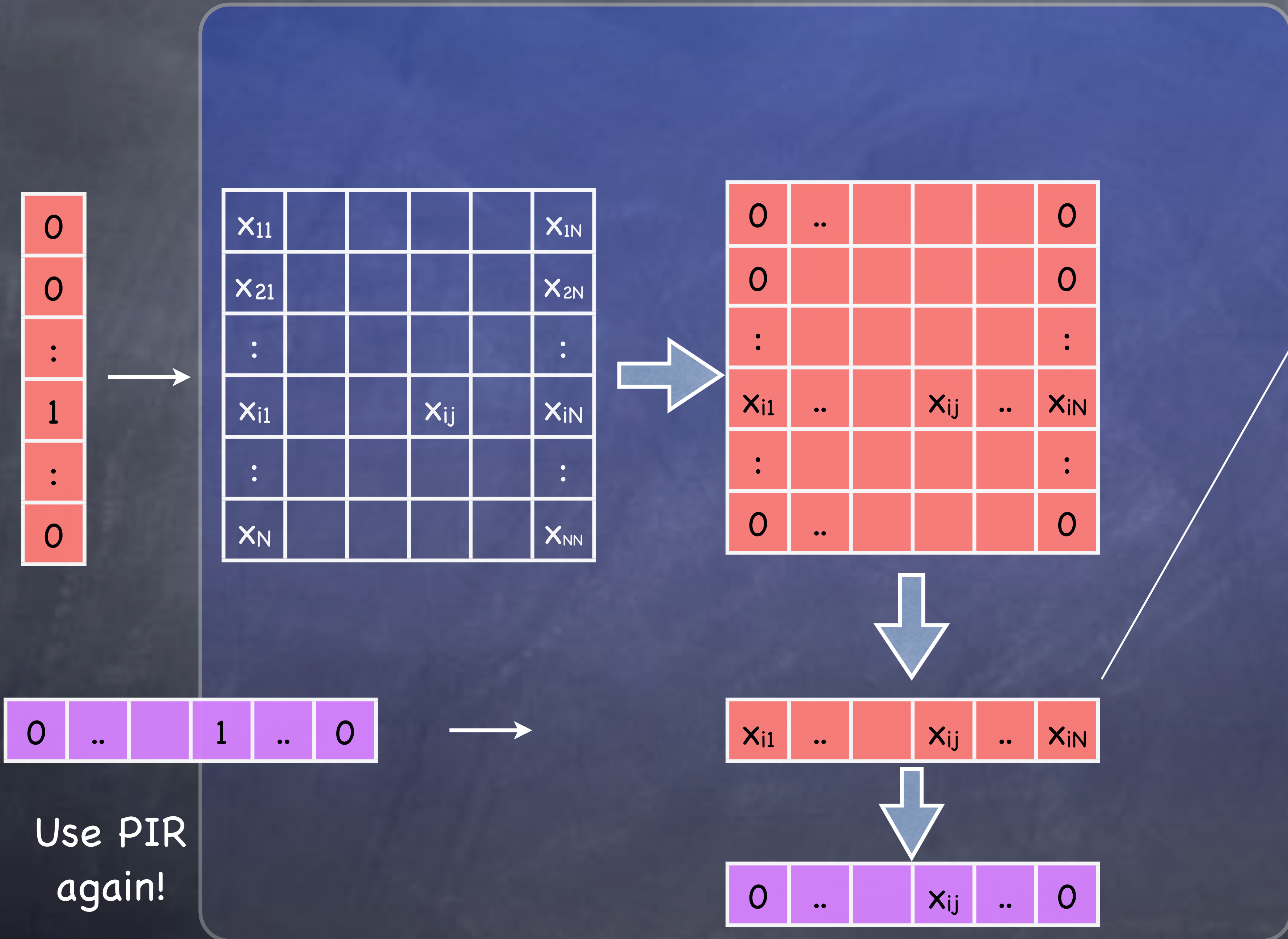

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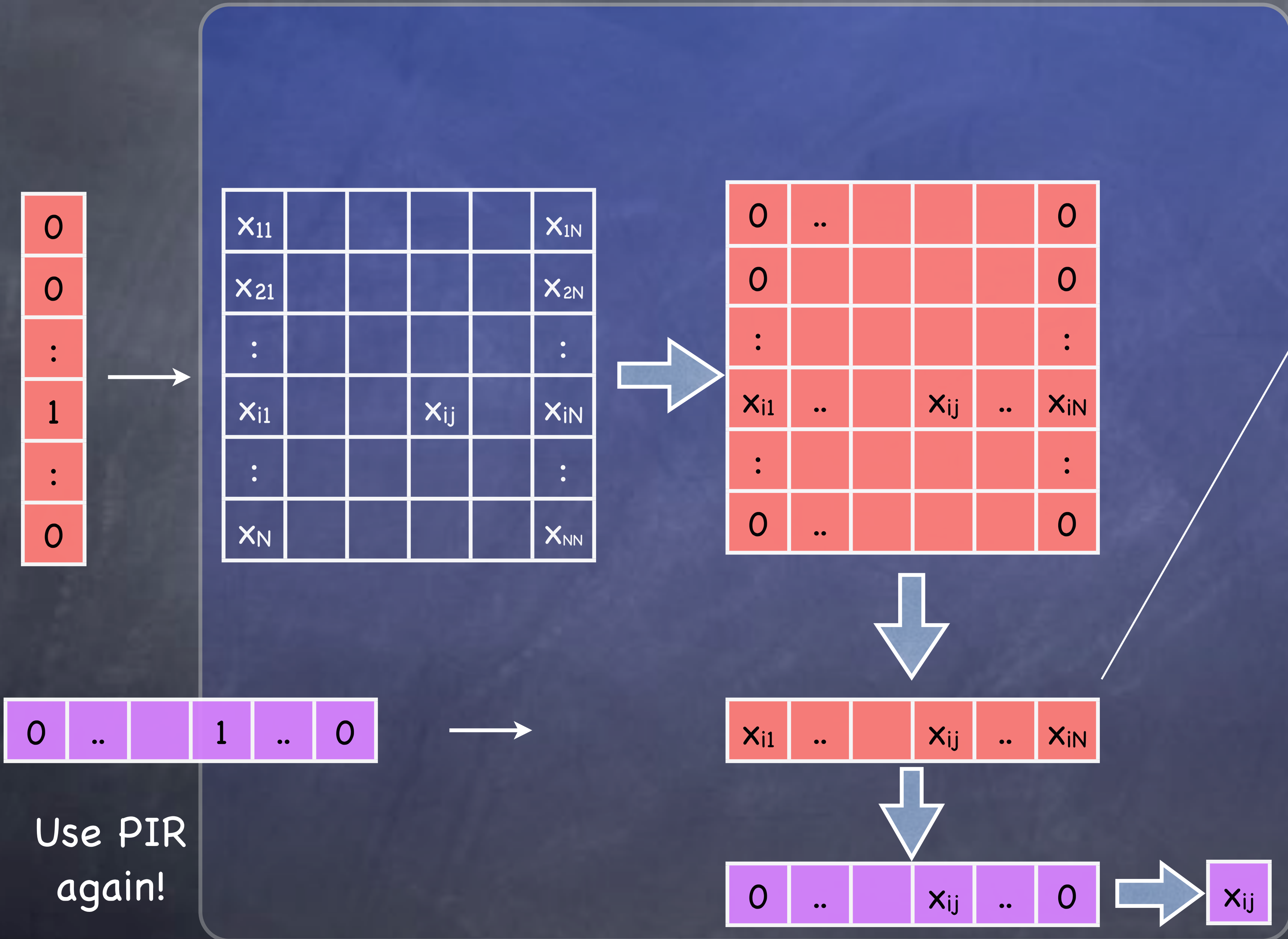

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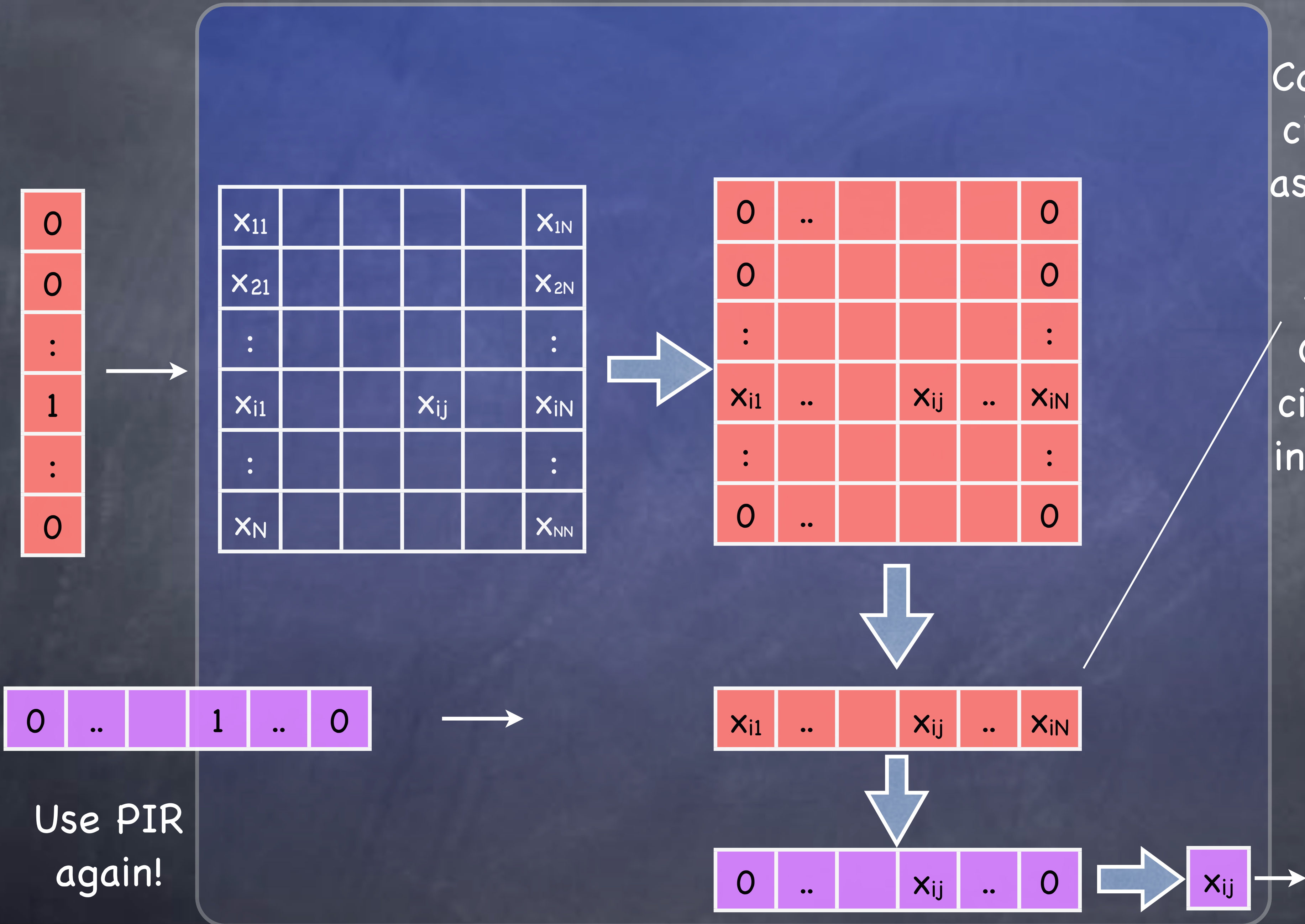

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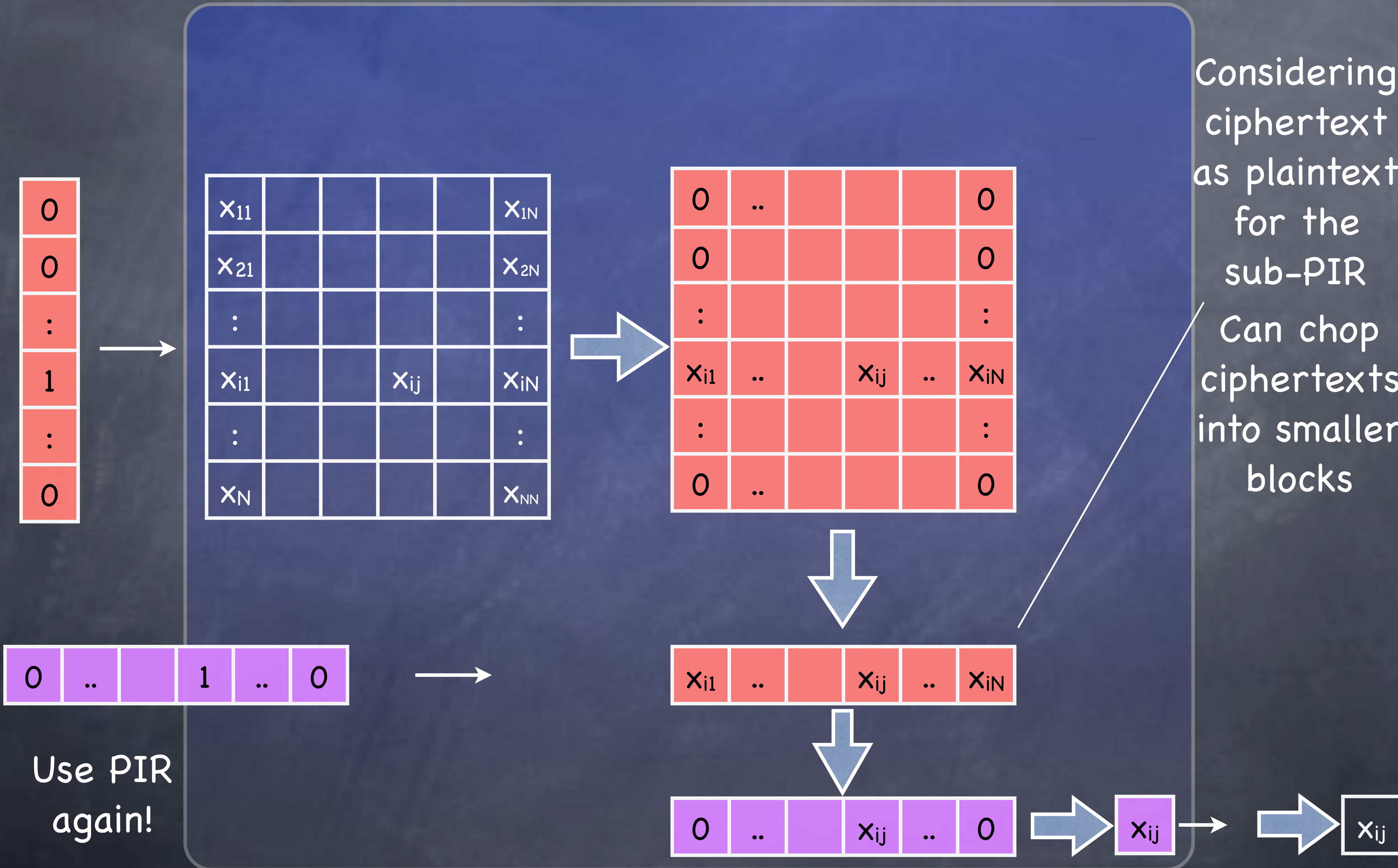

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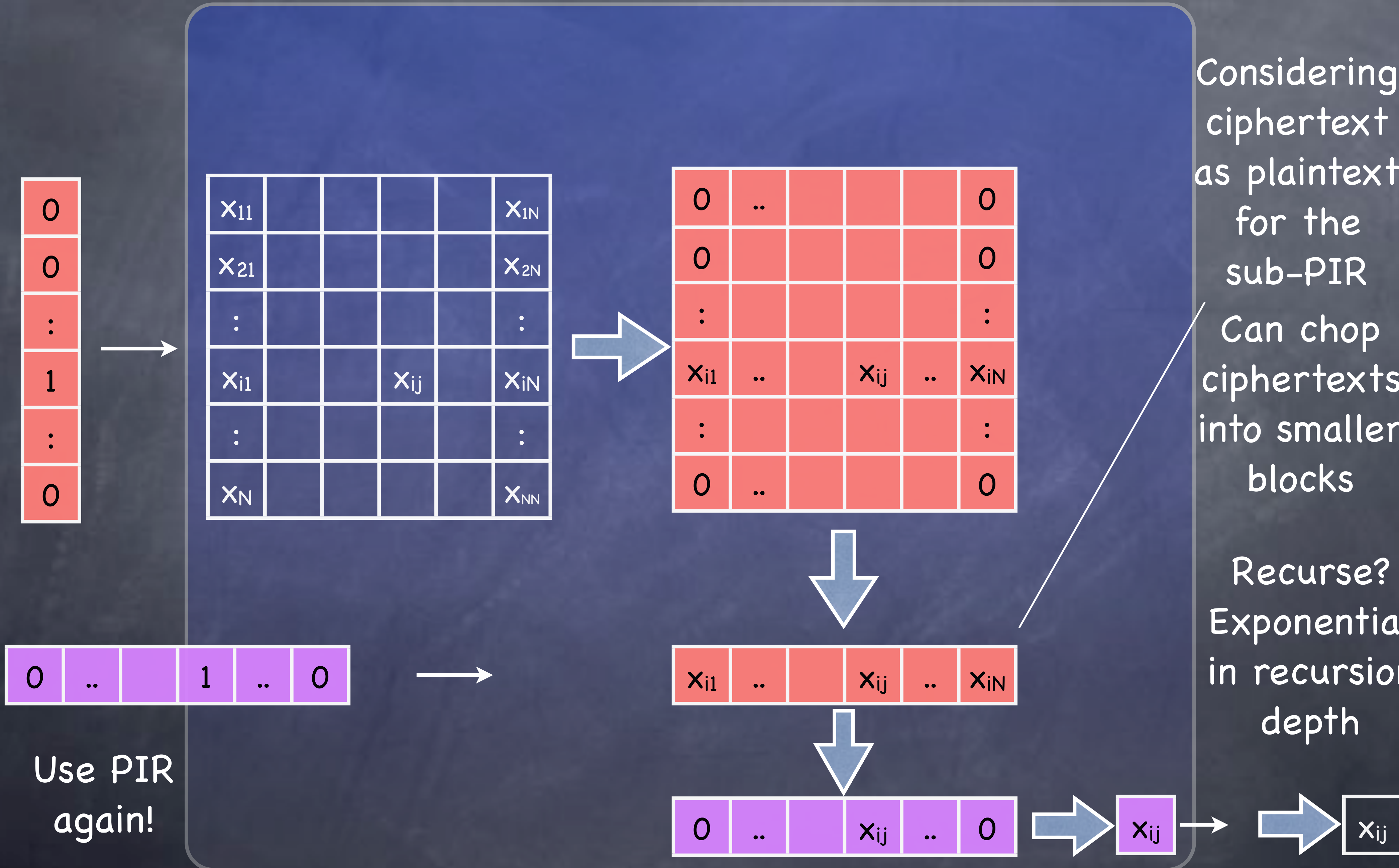

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Recurse? Exponential in recursion depth

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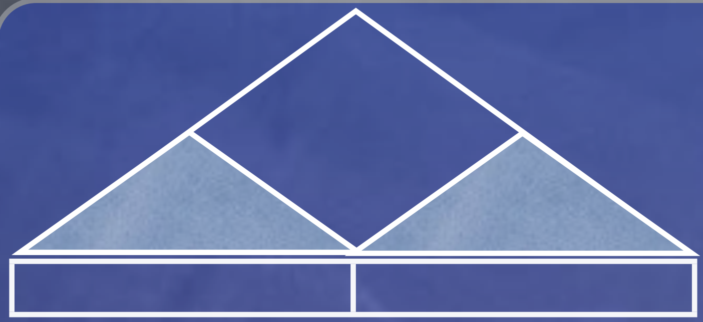
Final PIR protocol



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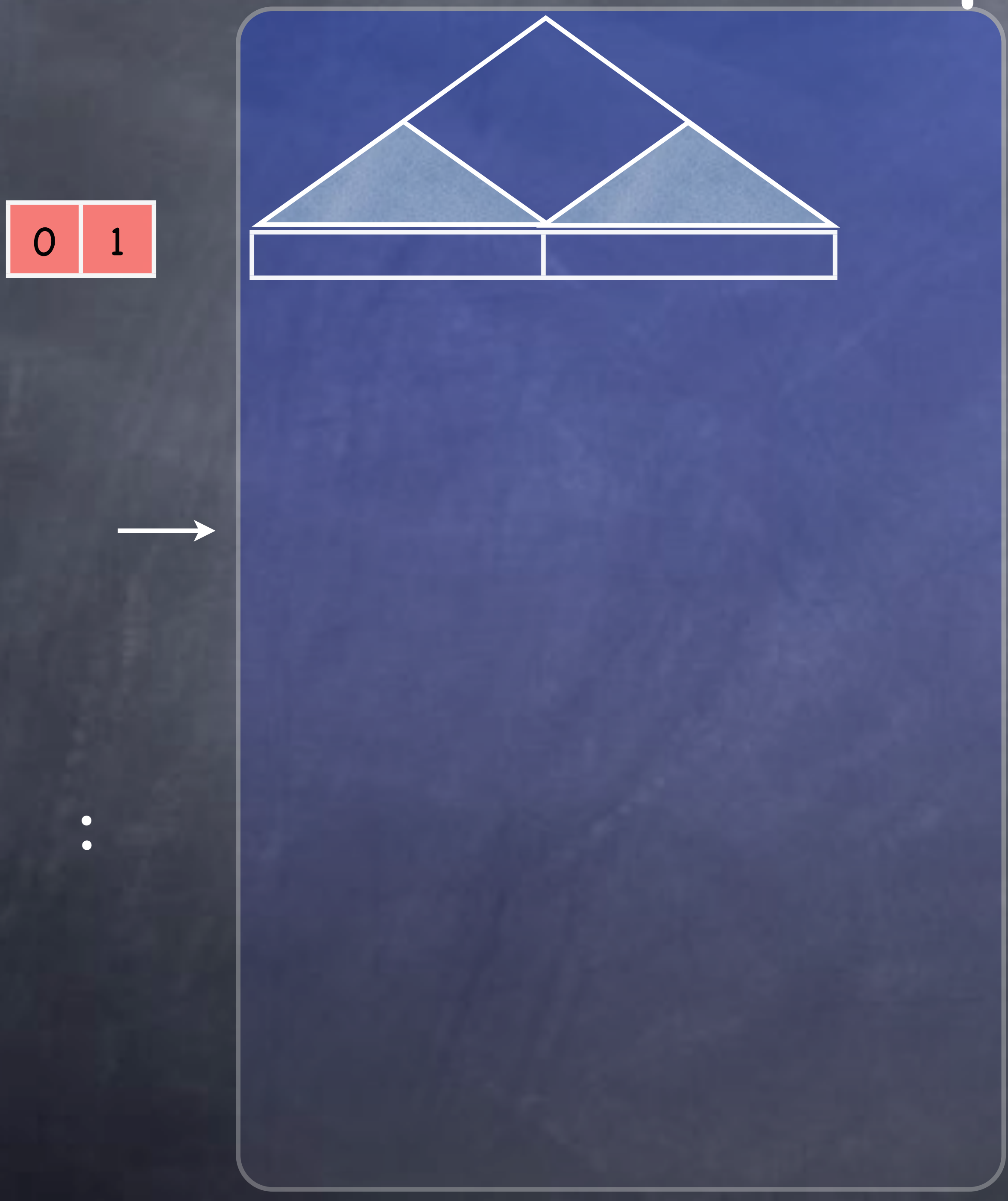

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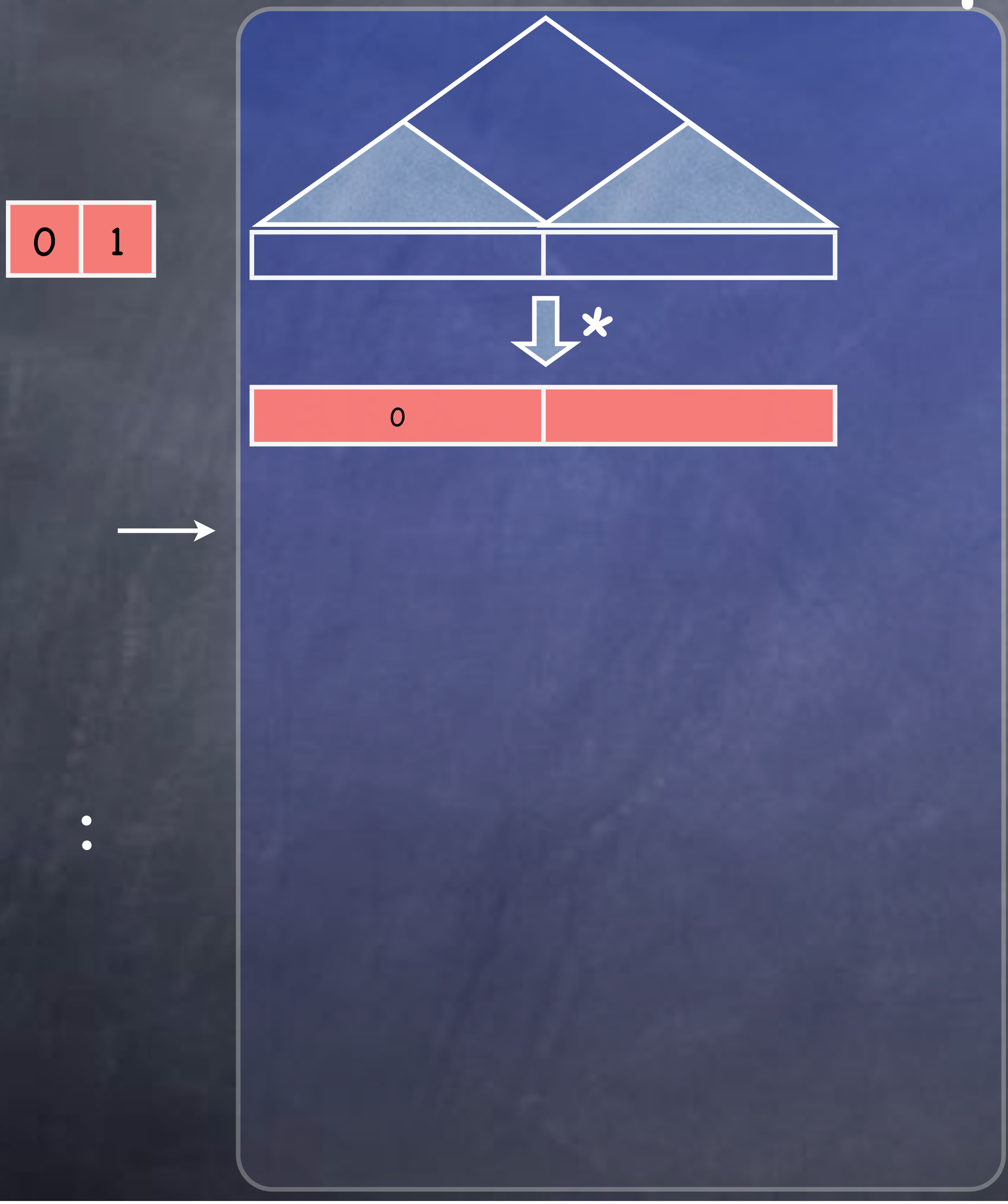

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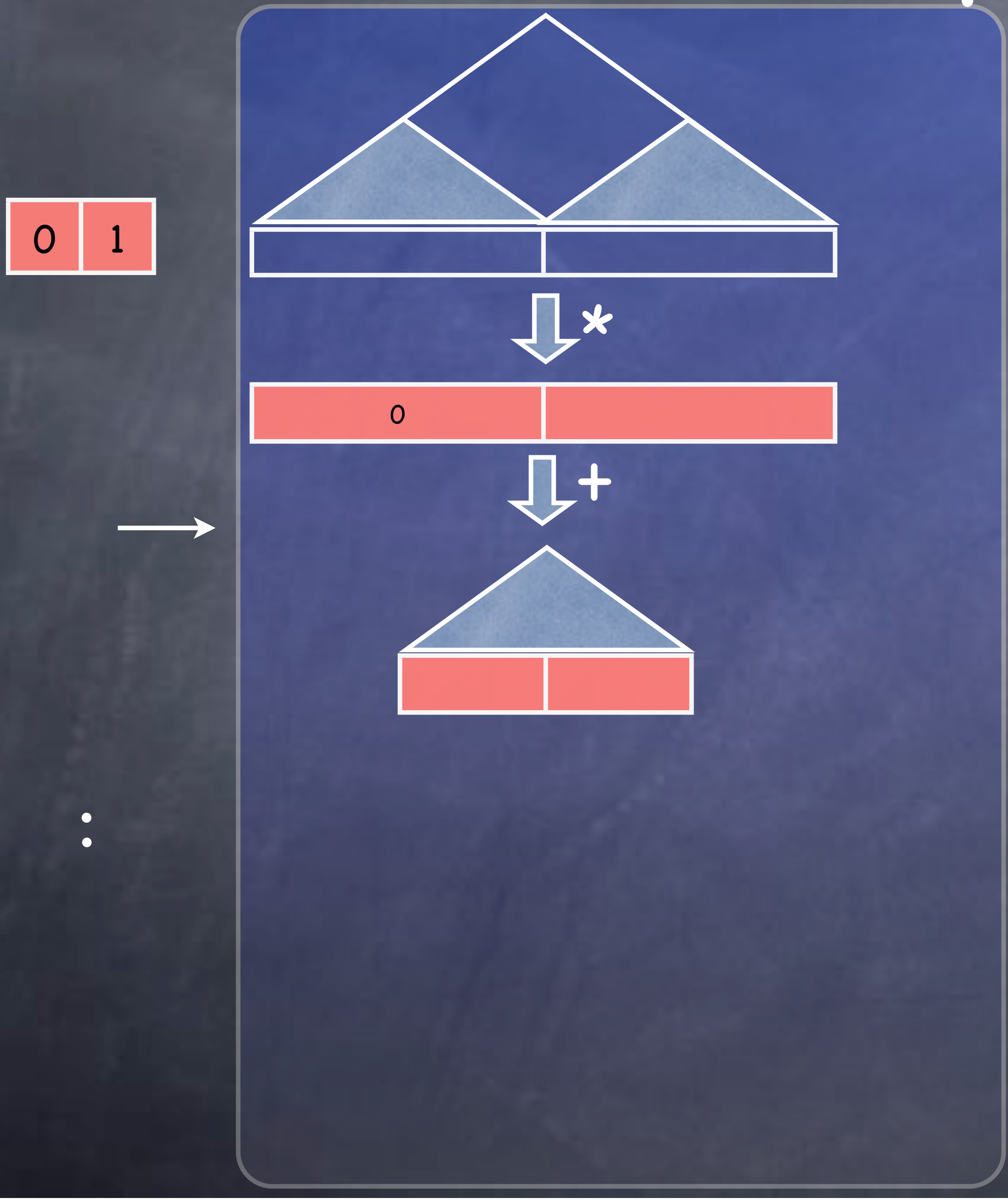

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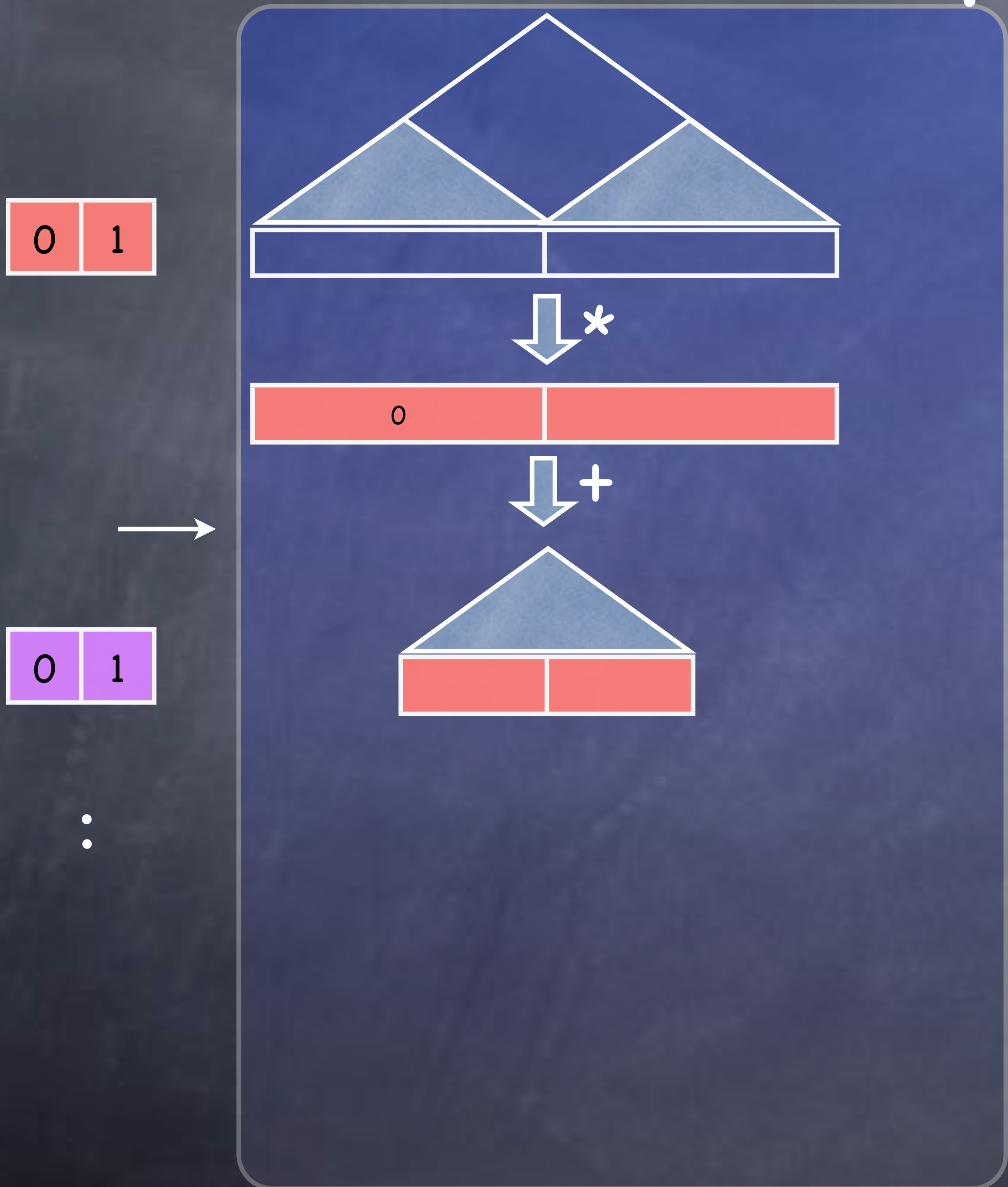

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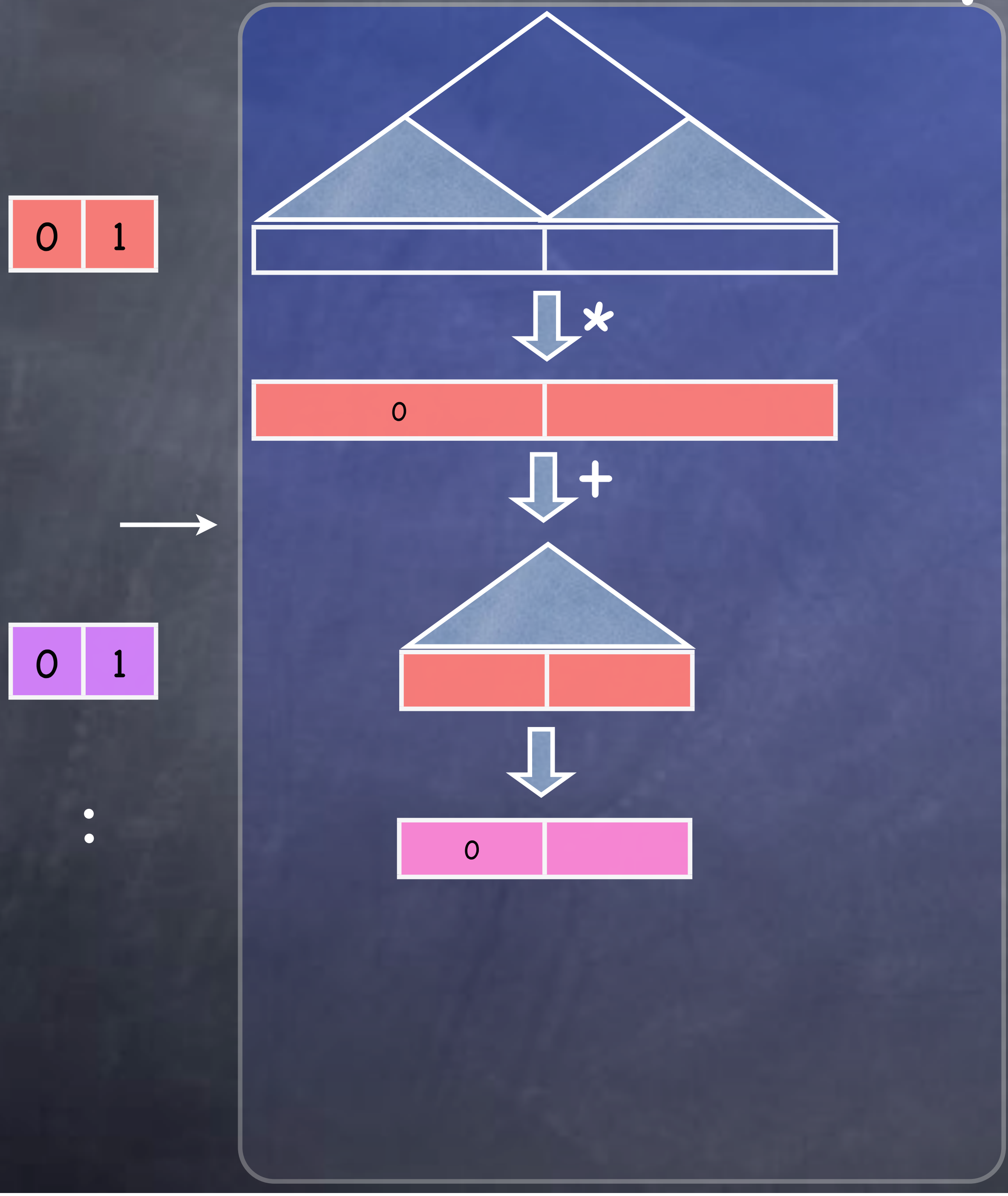

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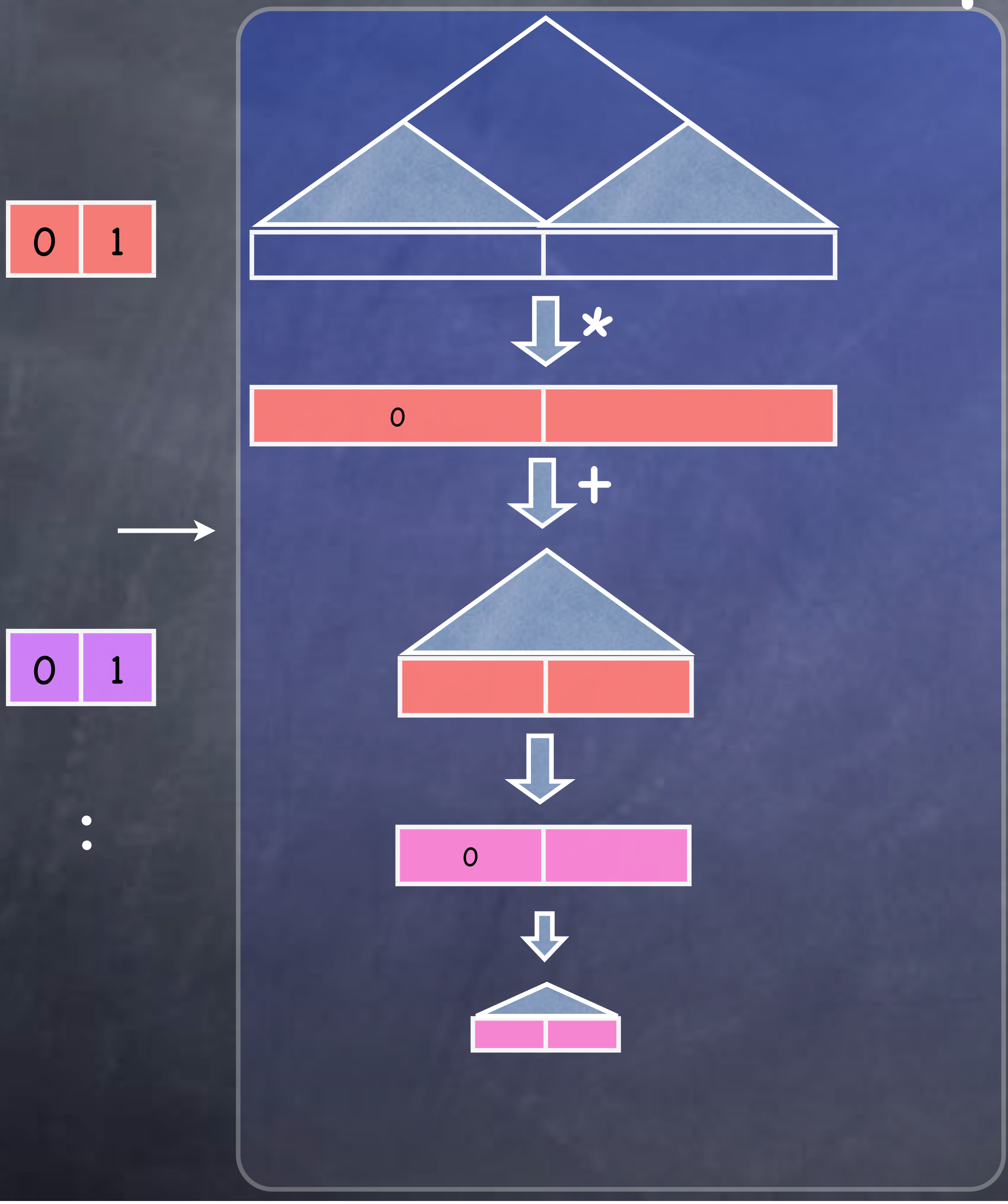

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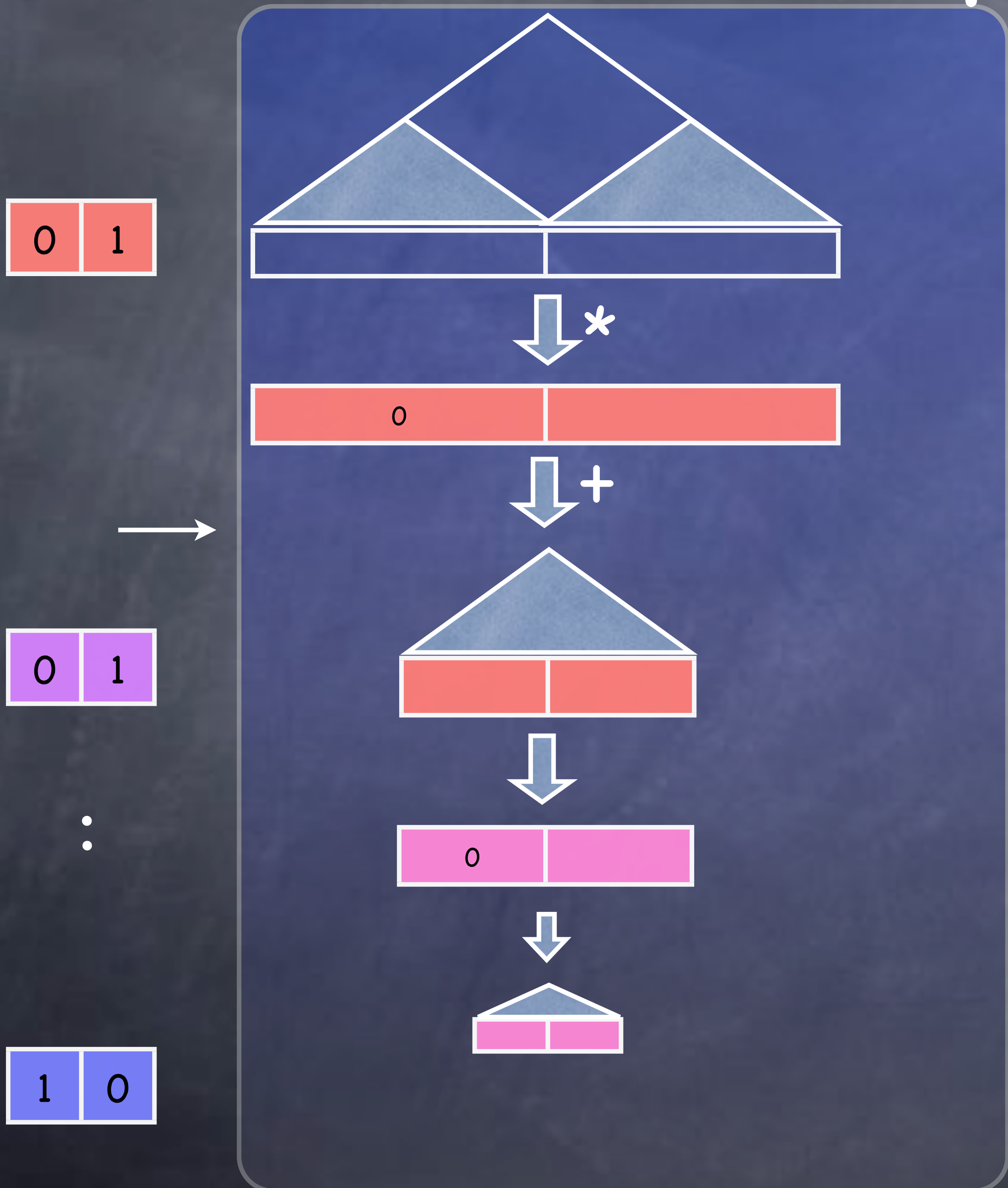

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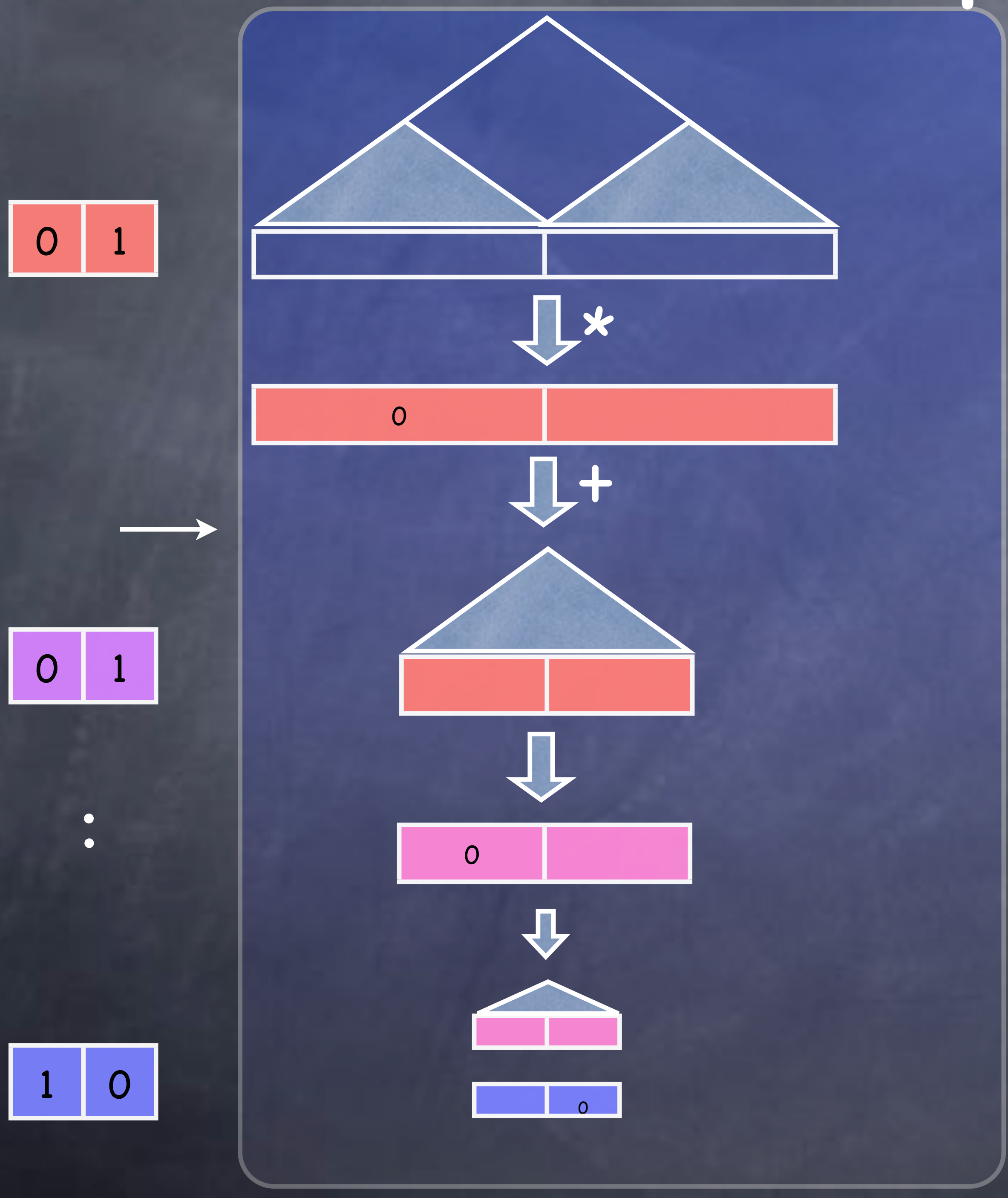

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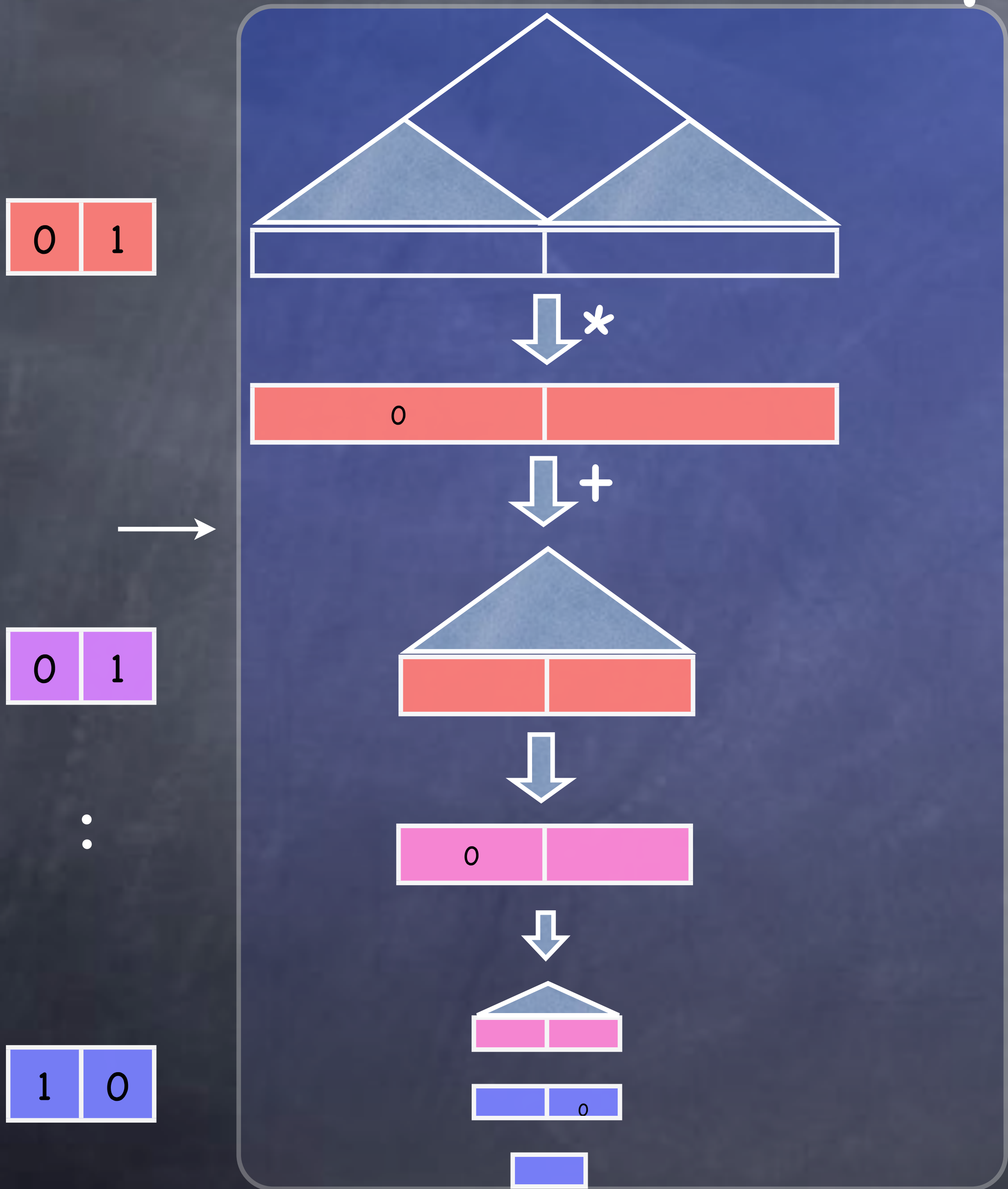

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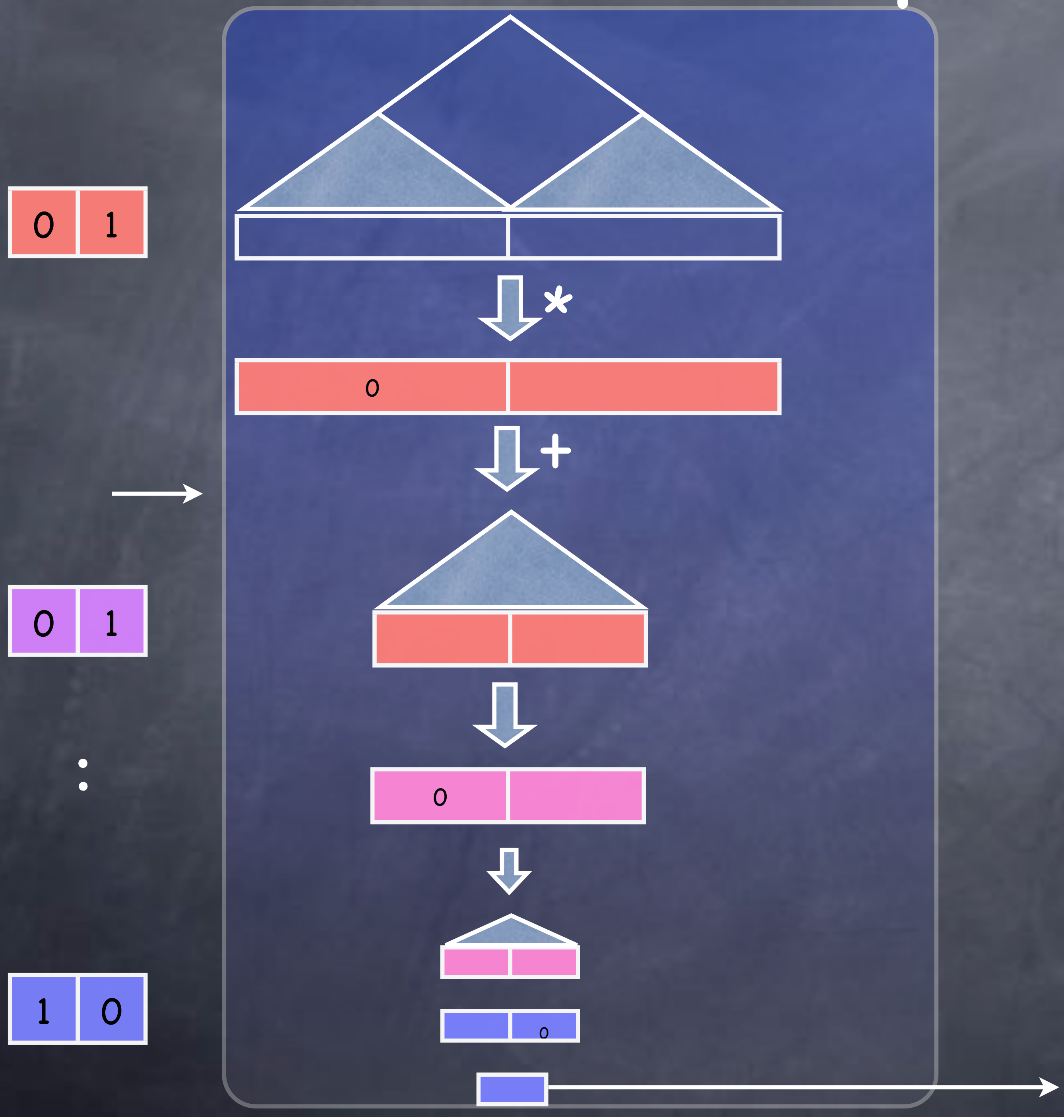

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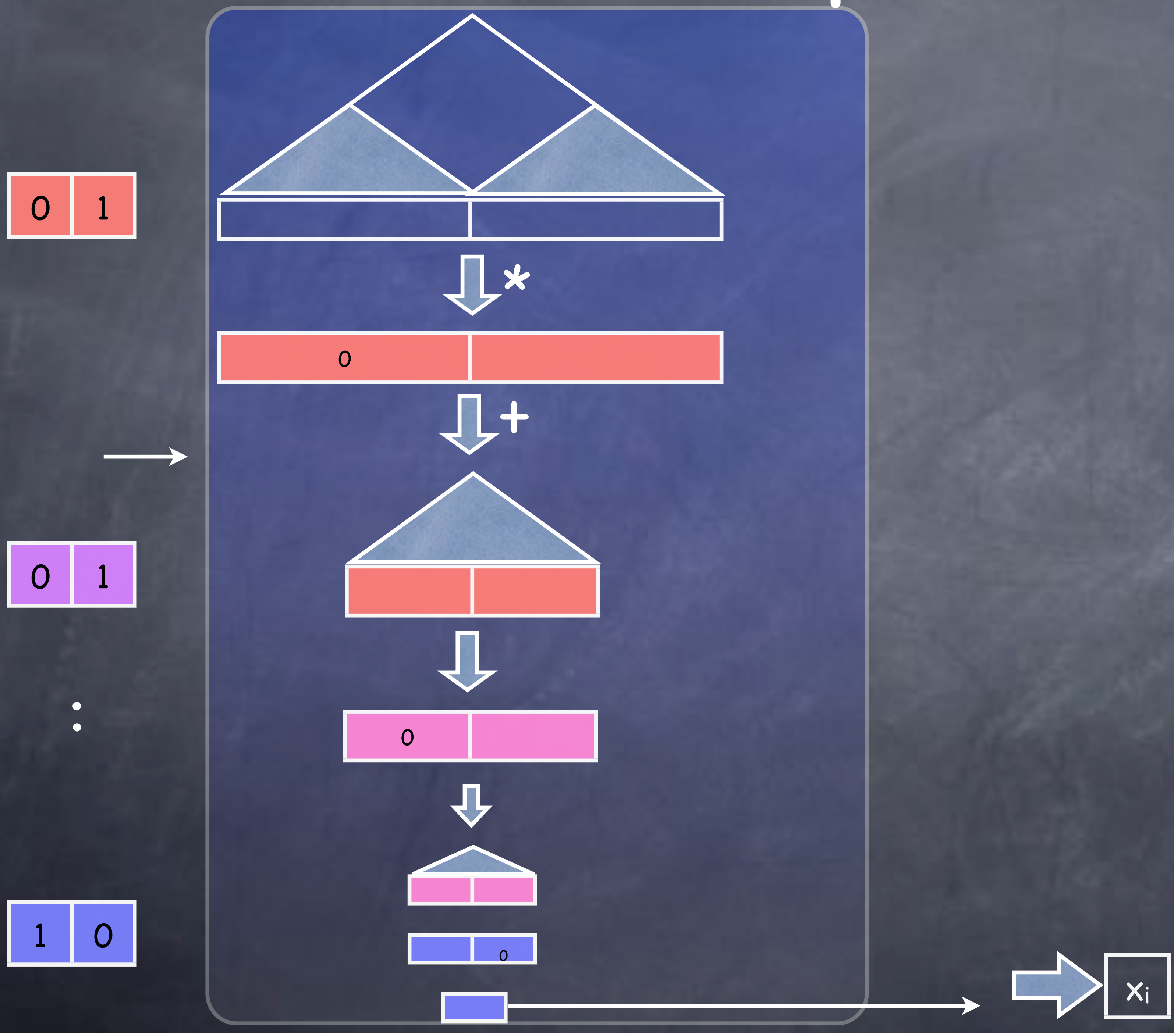

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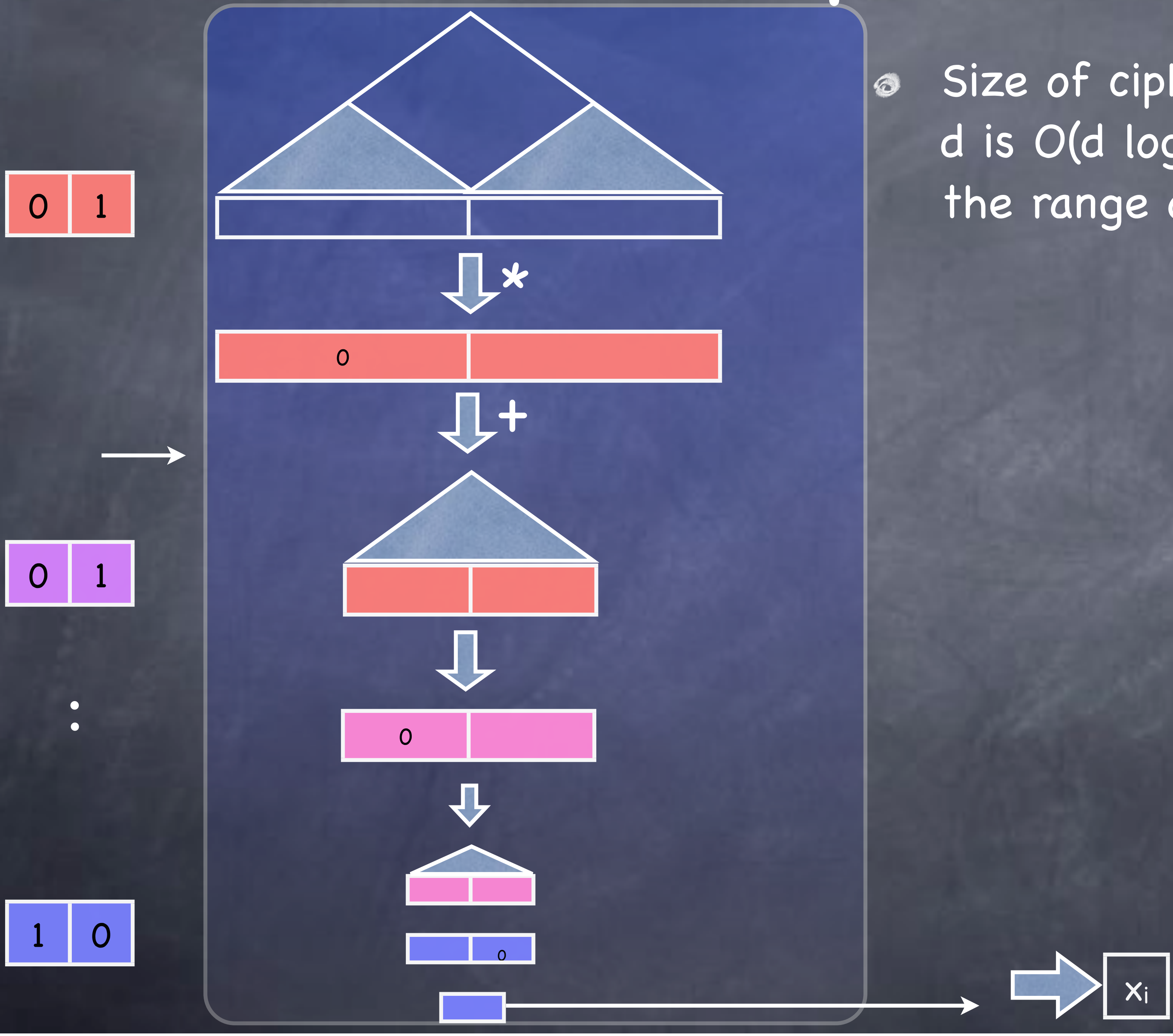

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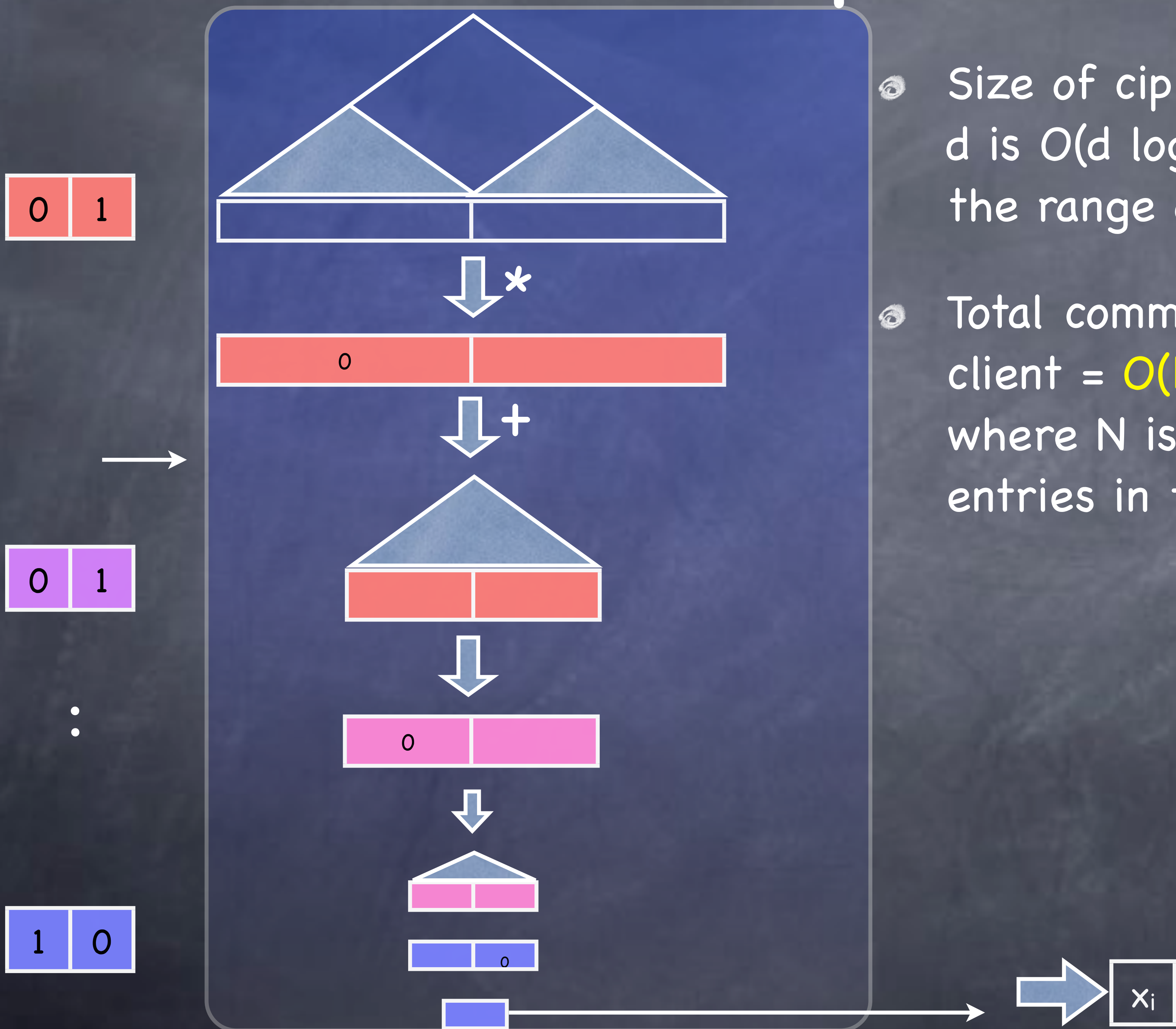

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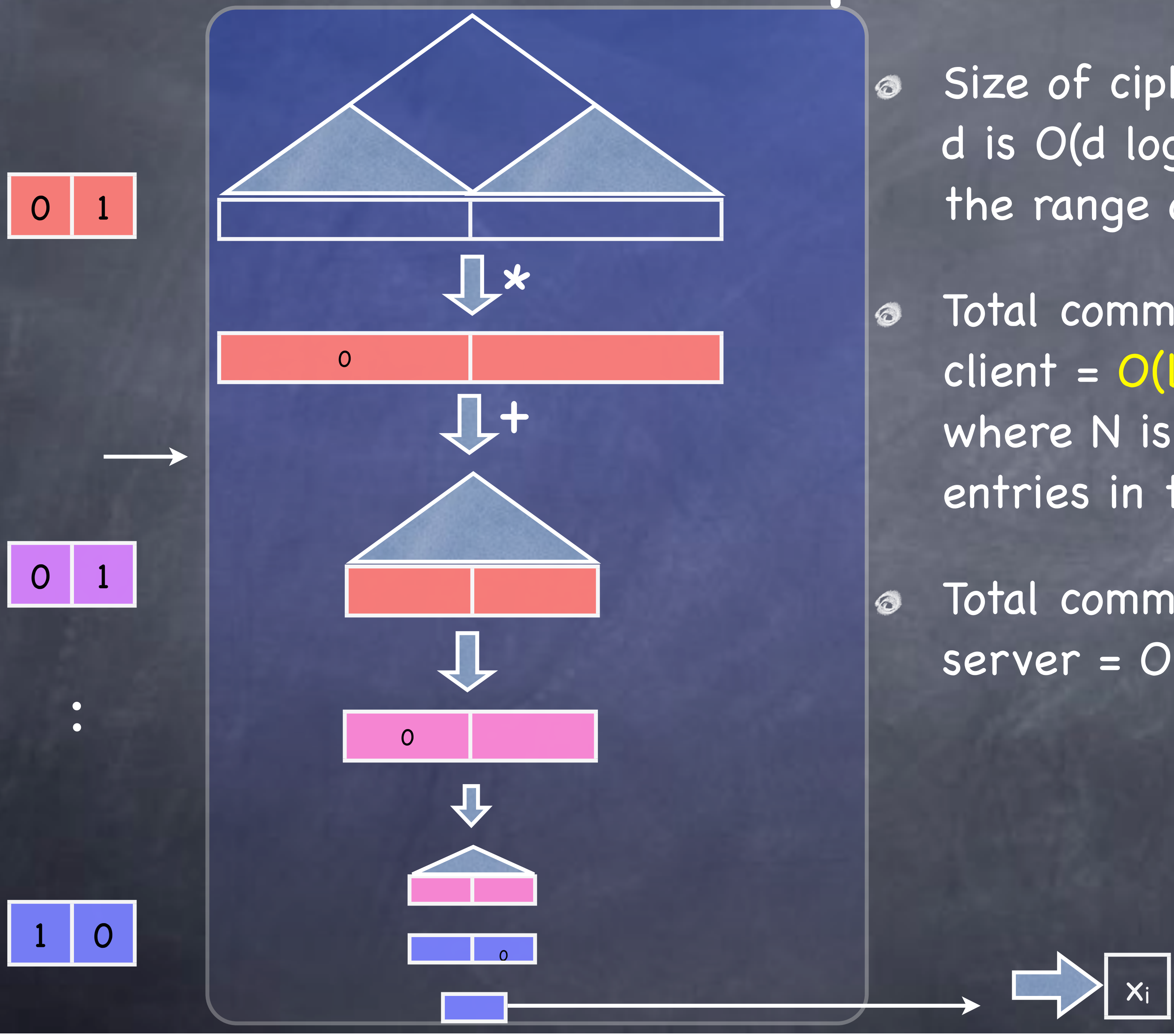

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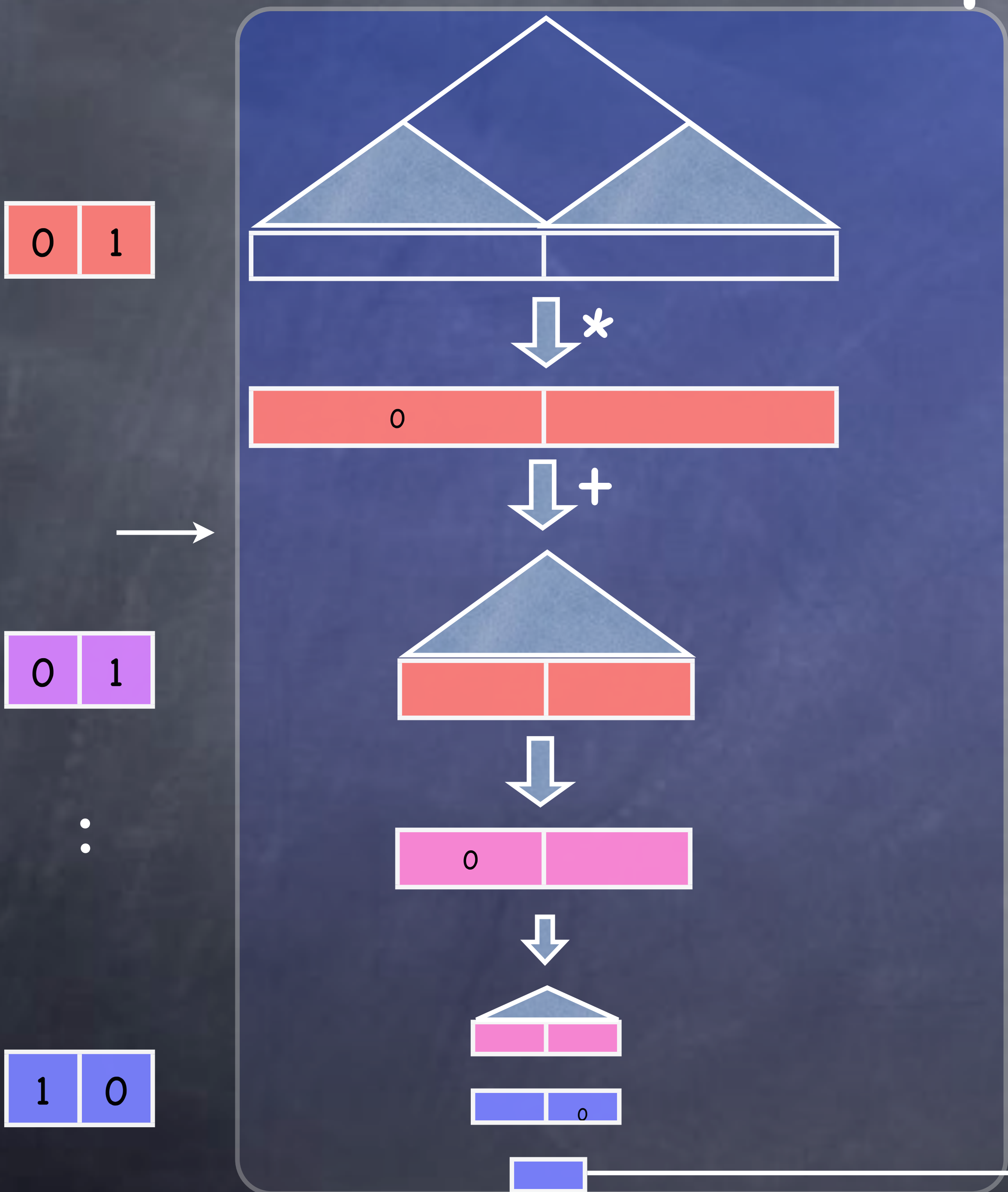

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  - Share x: All parties except P_1, choose their shares s_i; to help P_1 compute s_1, they publish $[-s_i]$, P_1 publishes [r]; they threshold decrypt $t=[r + x + \Sigma_{i=2:m} (-s_i)]$. P_1 sets $s_1 = t-r$
  - Each party publishes $s_i*[y] = [s_i\ y]$; they compute $[\Sigma s_i y]=[xy]$
- Threshold decrypt the output

For active-security, include ZK proofs of correctness/knowledge of plaintext, when publishing

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- For non-homomorphic encryption, not critical: can use a scheme with a larger domain into which the required domain can be embedded
- But not good for homomorphic encryption: say, an application needs to use addition modulo 10; can we use Paillier?

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    - Each time add a large random multiple of 10 (but not large enough to cause overflow): 9+3+10r and 2+10r are statistically close if r drawn from a large range

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- Coming up: more applications - in voting