Indian Buffet Process

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- Mixture Model
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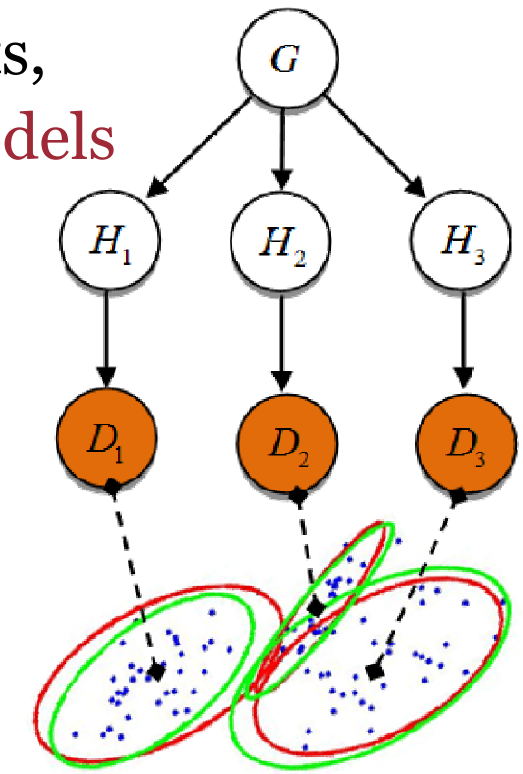
Bayesian Nonparametrics

- Models with undefined number of elements,
  - Dirichlet Process for infinite mixture models
  - With various applications
    - Hierarchies
    - Topics and syntactic classes
    - Objects appearing in one image



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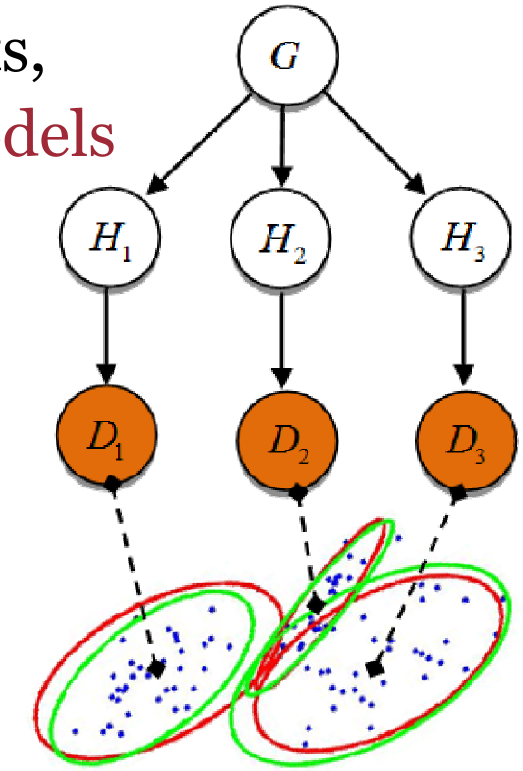
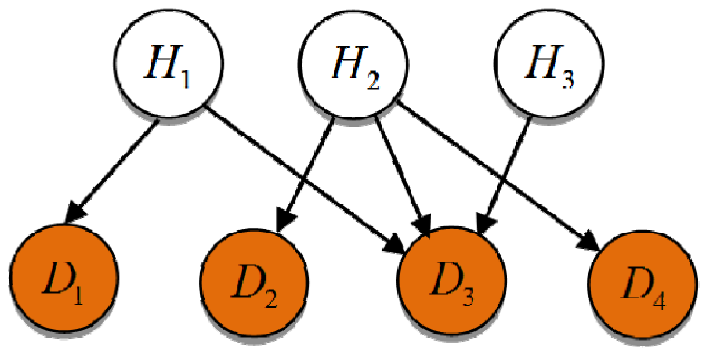


  - Cons
    - The models are limited to the case that could be modeled using DP.
    - i.e. set of observations are generated by only one latent component

Bayesian Nonparametrics contd.

- In practice there might be more complicated interaction between latent variables and observations



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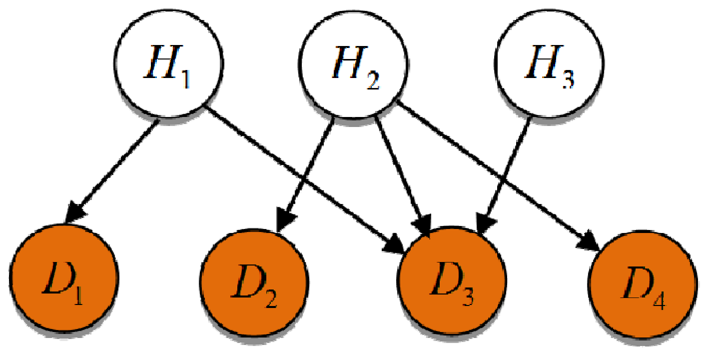

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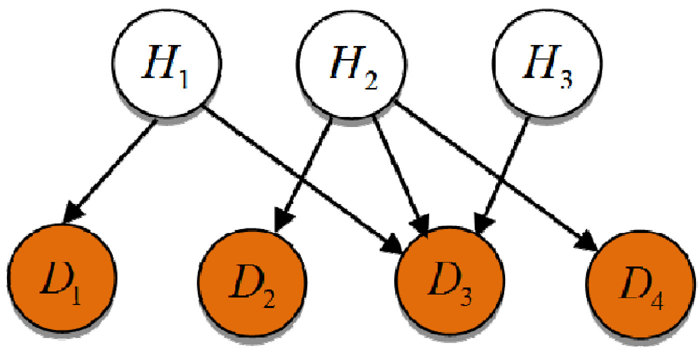

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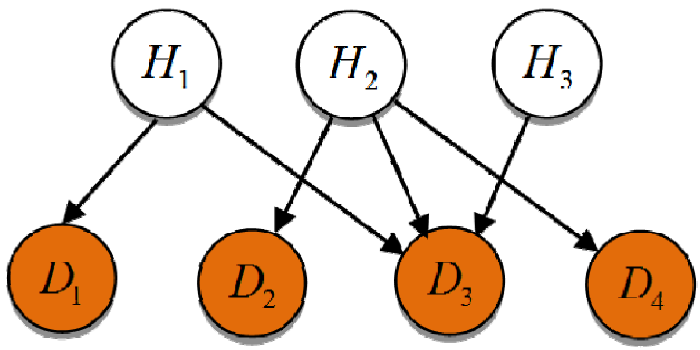

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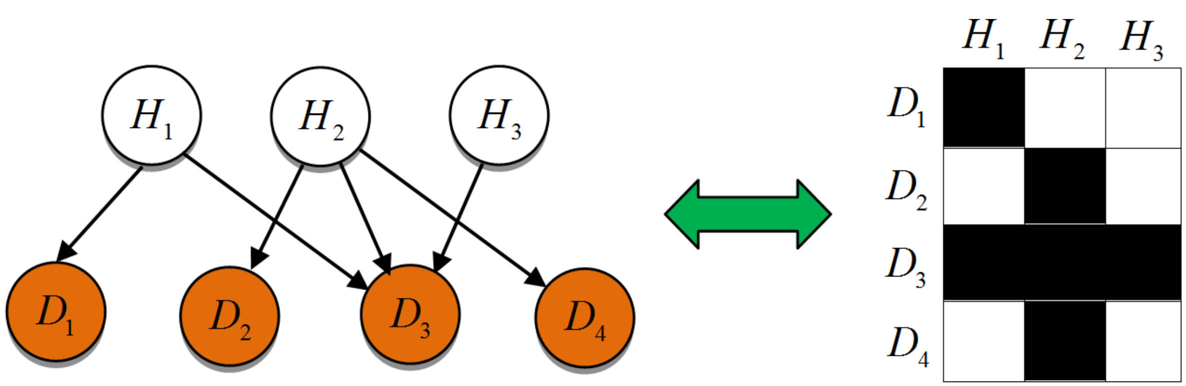
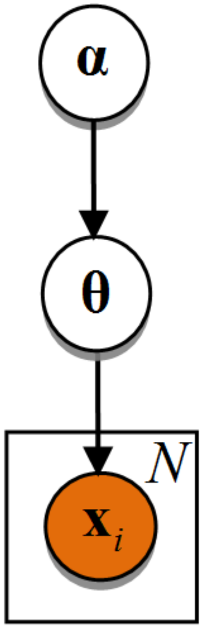


Finite Mixture Model

- Set of observation: $\{\mathbf{x}_i\}_{i=1}^{N}$



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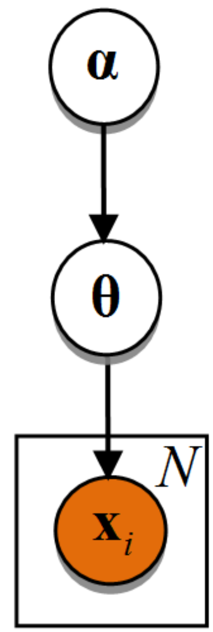

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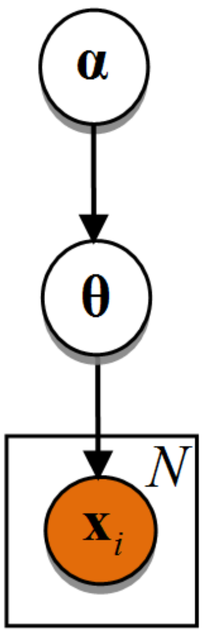

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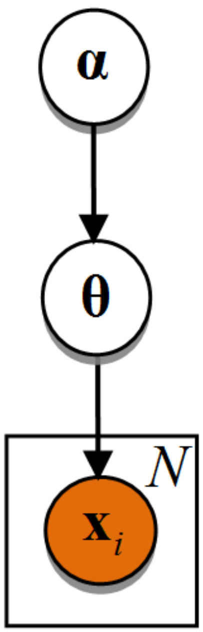

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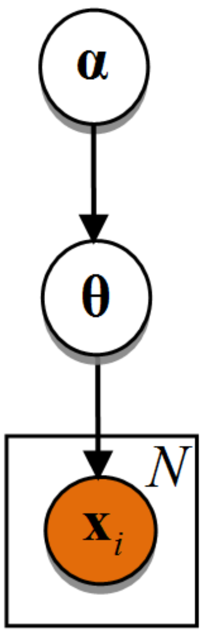

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- The probability of each sample under the model:

$$p(\mathbf{x}_i \mid \theta) = \sum_{k=1}^{K} p(\mathbf{x}_i \mid c_i = k)\, p(c_i = k)$$

- The likelihood of samples:

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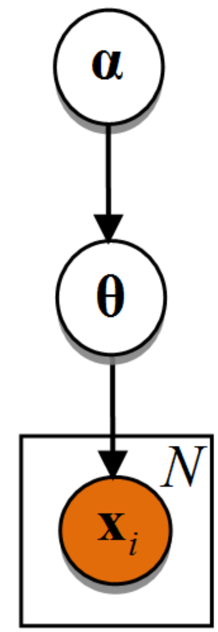

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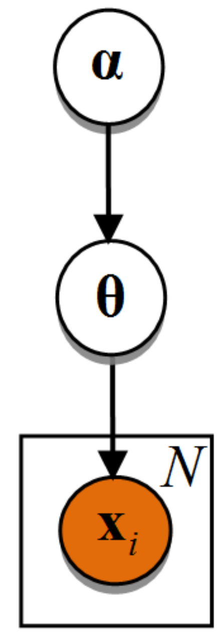
- The prior on the component probabilities (symmetric Dirichlet dits.)

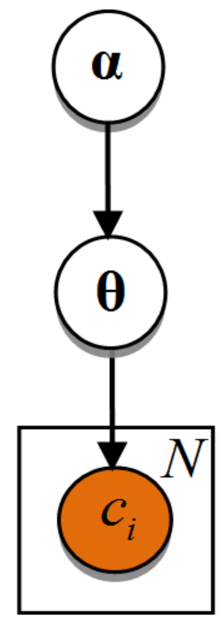
$$\theta \mid \alpha \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right).$$



Finite Mixture Model

- Since we want the mixture model to be valid for any general component $p(x_j \mid c_j = i)$ we only assume the number of cluster assignments to be the goal of learning this mixture model !
- Cluster assignments: $\mathbf{c} = [c_1, c_2, \ldots, c_N]^T$
- The model can be summarized as :

$$\begin{cases} \boldsymbol{\theta} \mid \boldsymbol{\alpha} \sim \text{Dirichlet}(\frac{\alpha}{K}, \ldots, \frac{\alpha}{K}). \\ c_i \mid \boldsymbol{\theta} \sim \text{Discrete}(\boldsymbol{\theta}) \end{cases}$$



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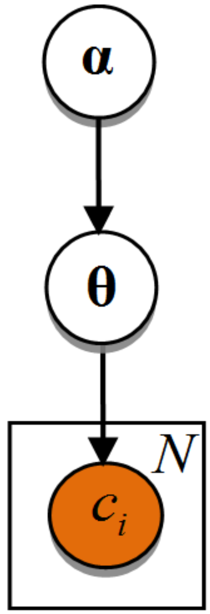

- To have a valid model, all of the distributions must be valid !

$$p(\boldsymbol{\theta} \mid \mathbf{c}) = \frac{p(\mathbf{c} \mid \boldsymbol{\theta}).p(\boldsymbol{\theta})}{p(\mathbf{c})}$$

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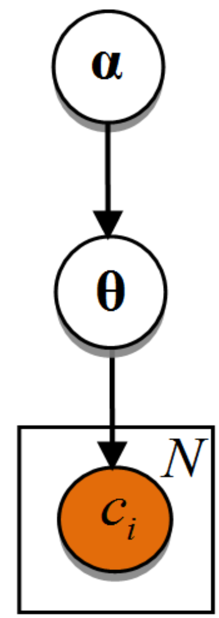

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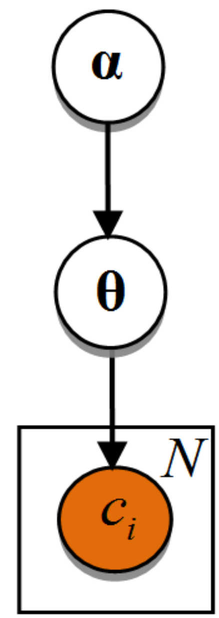

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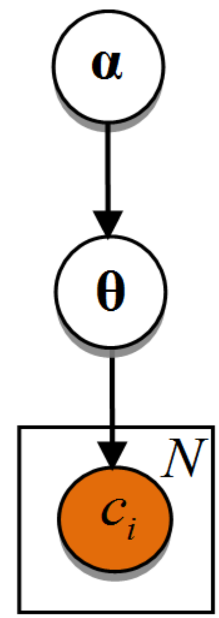

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posterior: $p(\boldsymbol{\theta} \mid \mathbf{c})$; **likelihood**: $p(\mathbf{c} \mid \boldsymbol{\theta})$; **prior**: $p(\boldsymbol{\theta})$

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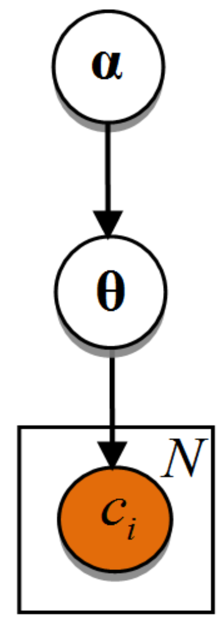

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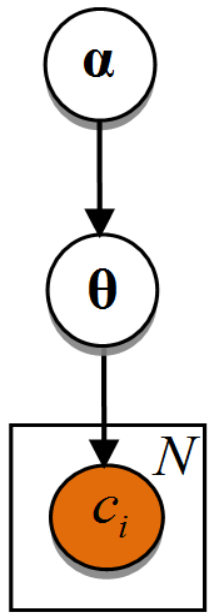
Marginal likelihood (Evidence): $p(\mathbf{c})$

Finite Mixture Model contd.

$$p(\mathbf{c}) = \int_{\Delta_K} p(\mathbf{c} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad p(\boldsymbol{\theta}) = \left(D(\frac{\alpha}{K}, \ldots, \frac{\alpha}{K}) \right)^{-1} \prod_{k=1}^{K} \theta_k^{\alpha_k - 1}$$



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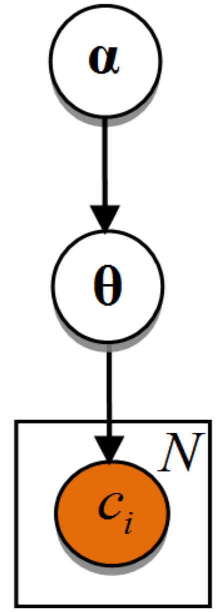

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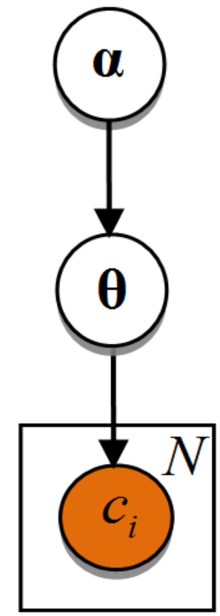

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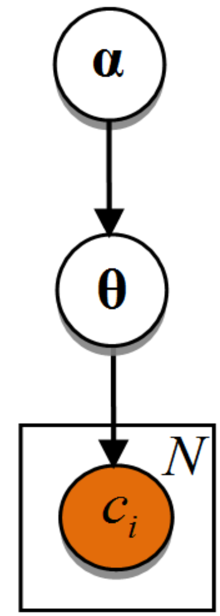

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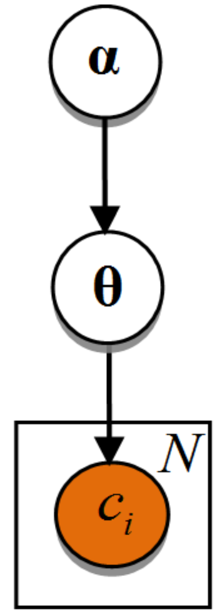

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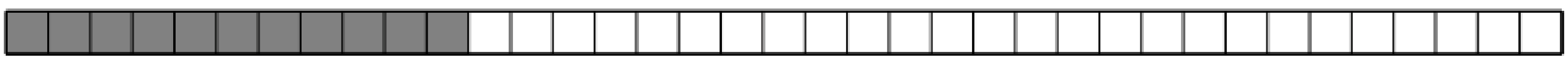
- Since we always have limited samples in reality, we will have limited number of clusters used; so we define two set of clusters:



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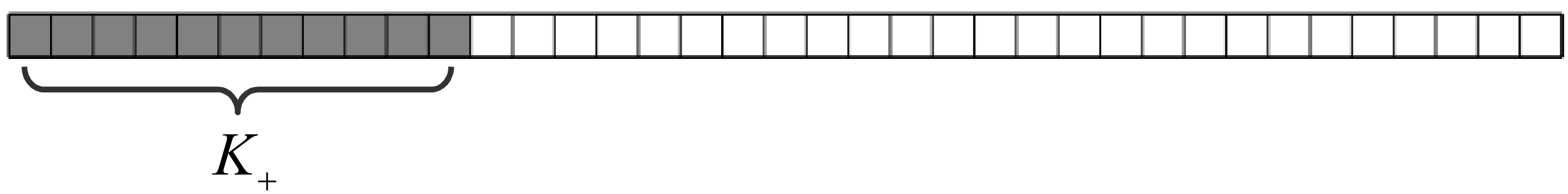

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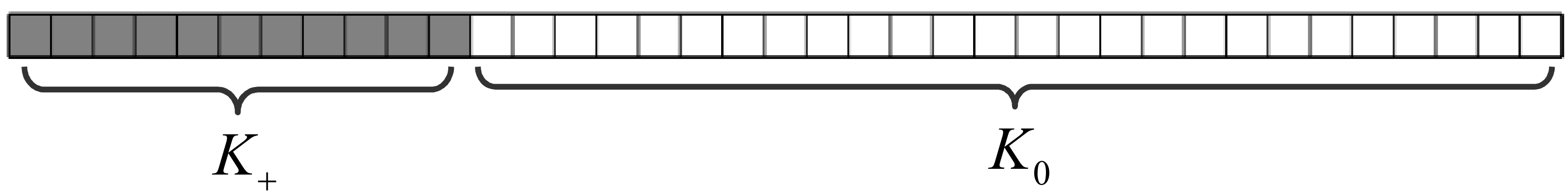

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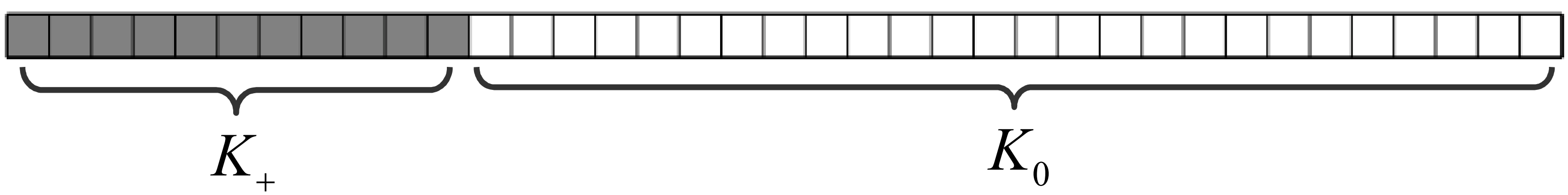

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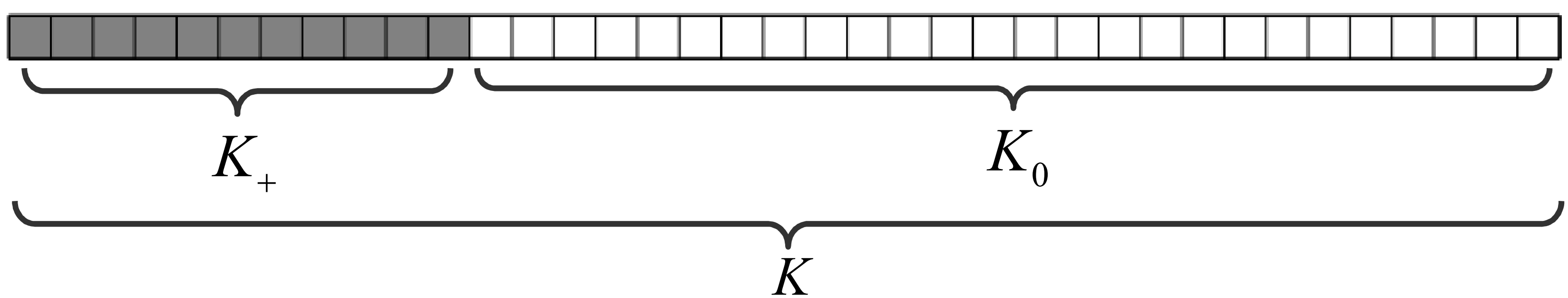

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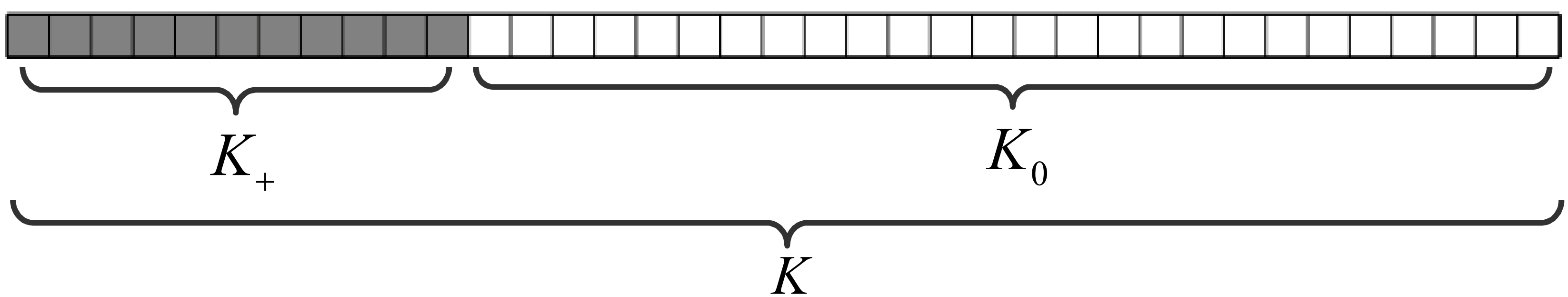

- Infinite clusters likelihood
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  - Infinite dimensional multinomial cluster distribution.

Infinite mixture model

- Now we return to the previous slides and set $K \to \infty$ in formulas

$$p(\mathbf{c}) = \frac{\prod_{k=1}^{K} \Gamma\left(m_k + \frac{\alpha}{K}\right)}{\left(\Gamma\left(\frac{\alpha}{K}\right)\right)^K} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}, \quad \text{s.t.} \quad m_k = \sum_{i=1}^{N} \delta(c_i = k)$$

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$$\Gamma(x+1) = x\Gamma(x) \Rightarrow \Gamma(m_k + \frac{\alpha}{K}) = (m_k + \frac{\alpha}{K} - 1)\ldots(\frac{\alpha}{K})\Gamma(\frac{\alpha}{K})$$

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$$\Gamma(x+1) = x\Gamma(x) \Rightarrow \Gamma(m_k + \frac{\alpha}{K}) = (m_k + \frac{\alpha}{K} - 1)\ldots(\frac{\alpha}{K})\Gamma(\frac{\alpha}{K}) \Rightarrow \frac{\Gamma(m_k + \frac{\alpha}{K})}{\Gamma(\frac{\alpha}{K})} = (\frac{\alpha}{K})\prod_{j=1}^{m_k-1}\left(j + \frac{\alpha}{K}\right)$$

Infinite mixture model

- Now we return to the previous slides and set $K \to \infty$ in formulas

$$p(\mathbf{c}) = \frac{\prod_{k=1}^{K} \Gamma\left(m_k + \frac{\alpha}{K}\right)}{\left(\Gamma\left(\frac{\alpha}{K}\right)\right)^K} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}, \quad \text{s.t.} \quad m_k = \sum_{i=1}^{N} \delta(c_i = k)$$

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$$\Rightarrow p(\mathbf{c}) = \frac{\prod_{k=1}^{K} \Gamma(m_k + \frac{\alpha}{K})}{\left(\Gamma(\frac{\alpha}{K})\right)^K} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} = \left(\frac{\alpha}{K}\right)^{K_+} \prod_{k=1}^{K_+} \prod_{j=1}^{m_k-1} \left(j + \frac{\alpha}{K}\right) \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$$

Infinite mixture model

- Now we return to the previous slides and set $K \to \infty$ in formulas

$$p(\mathbf{c}) = \frac{\prod_{k=1}^{K} \Gamma\left(m_k + \frac{\alpha}{K}\right)}{\left(\Gamma\left(\frac{\alpha}{K}\right)\right)^K} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}, \quad \text{s.t.} \quad m_k = \sum_{i=1}^{N} \delta(c_i = k)$$

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If we set $K \to \infty$ the marginal likelihood will be $p(\mathbf{c}) \to 0$.
Instead we can model this problem, by defining probabilities on **partitions of samples**, instead of **class labels for each sample**.

Infinite mixture model contd.

- Define a partition of objects;
- Want to partition N objects into K_+ classes

Infinite mixture model contd.

- Define a partition of objects;
- Want to partition N objects into K_+ classes
- Equivalence class of object partitions: $[\mathbf{c}] = \{\mathbf{c}_i \mid \mathbf{c}_i \in \mathbf{c}\}$

Infinite mixture model contd.

- Define a partition of objects;
- Want to partition N objects into K_+ classes
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$$p([\mathbf{c}]) = \sum_{\mathbf{c}\in[\mathbf{c}]} p(\mathbf{c}) = \frac{K!}{K_0!}\left(\frac{\alpha}{K}\right)^{K_+} \prod_{k=1}^{K_+} \prod_{j=1}^{m_k-1}\left(j+\frac{\alpha}{K}\right)\frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$$

Infinite mixture model contd.

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$$\Rightarrow p([\mathbf{c}]) = \alpha^{K_+} . \frac{K!}{K_0! K^{K_+}} . \prod_{k=1}^{K_+} \prod_{j=1}^{m_k-1} \left(j + \frac{\alpha}{K}\right) . \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$$

Infinite mixture model contd.

- Define a partition of objects;
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Infinite mixture model contd.

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$$\Rightarrow \lim_{K\to\infty} p([\mathbf{c}]) = \alpha^{K_+}. \quad 1 \quad . \quad \prod_{k=1}^{K_+}(m_k-1)! \quad .\frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$$

Infinite mixture model contd.

- Define a partition of objects;
- Want to partition N objects into K_+ classes
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- Valid probability distribution for an infinite mixture model

Infinite mixture model contd.

- Define a partition of objects;
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- Valid probability distribution for an infinite mixture model
- Exchangeable with respect to clusters assignments !

Infinite mixture model contd.

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- Valid probability distribution for an infinite mixture model
- Exchangeable with respect to clusters assignments !
  - Important for Gibbs sampling (and Chinese restaurant process)

Infinite mixture model contd.

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$$\Rightarrow \lim_{K\to\infty} p([\mathbf{c}]) = \alpha^{K_+}.\quad 1 \quad .\quad \prod_{k=1}^{K_+}(m_k-1)! \quad .\frac{\Gamma(\alpha)}{\Gamma(N+\alpha)}$$

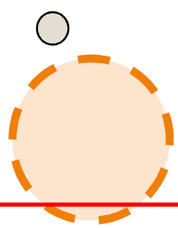
- Valid probability distribution for an infinite mixture model
- Exchangeable with respect to clusters assignments !
  - Important for Gibbs sampling (and Chinese restaurant process)
  - Di Finetti's theorem: explains why **exchangeable** observations are **conditionally independent** given some **probability distribution**

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 0$



$$p(c_1 = 1) = \frac{1}{1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 1,$ $m_2 = 0$

$$p(c_2 = 1 \mid c_1) = \frac{1}{1+1}$$

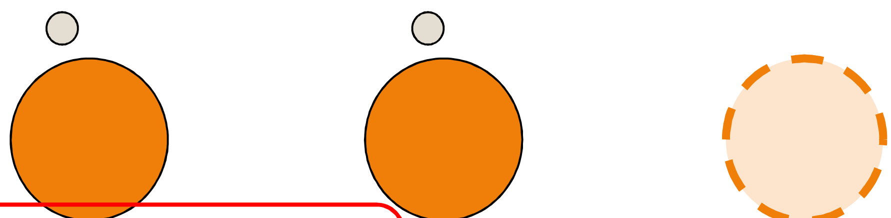
$$p(c_2 = 2 \mid c_1) = \frac{1}{1+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 1,$ $m_2 = 1,$ $m_3 = 0$



$$p(c_3 = 1 \mid c_{1:2}) = \frac{1}{2+1}$$

$$p(c_3 = 3 \mid c_{1:2}) = \frac{1}{2+1}$$

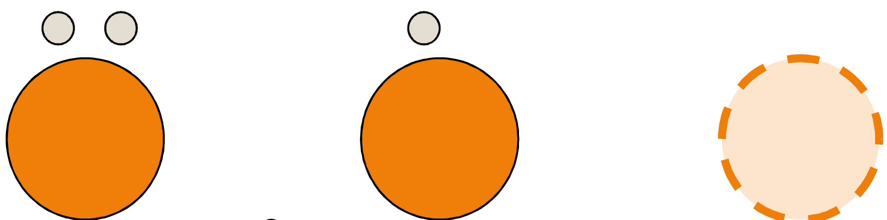
$$p(c_3 = 2 \mid c_{1:2}) = \frac{1}{2+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 2,$ $m_2 = 1,$ $m_3 = 0$



$$p(c_4 = 1 \mid c_{1:3}) = \frac{2}{3+1}$$

$$p(c_4 = 3 \mid c_{1:3}) = \frac{1}{3+1}$$

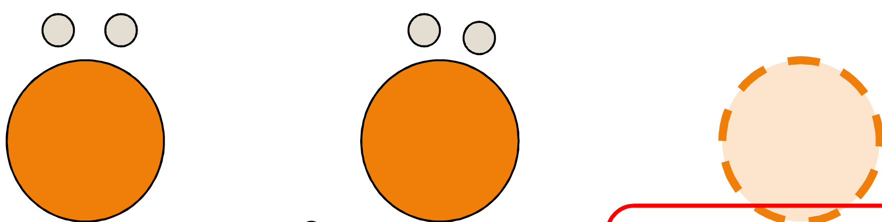
$$p(c_4 = 2 \mid c_{1:3}) = \frac{1}{3+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 2,$ $m_2 = 2,$ $m_3 = 0$



$$p(c_5 = 1 \mid c_{1:4}) = \frac{2}{4+1}$$

$$p(c_5 = 3 \mid c_{1:4}) = \frac{1}{4+1}$$

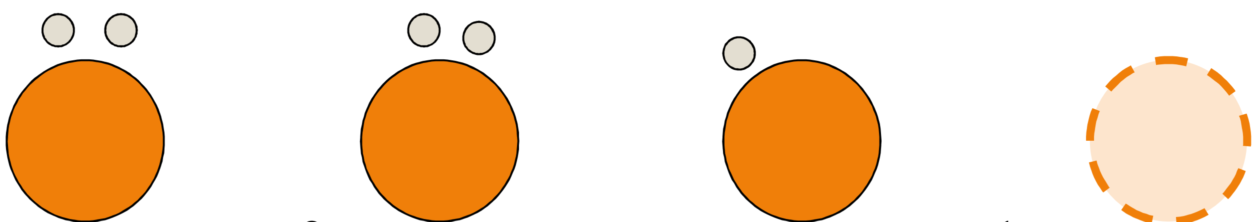
$$p(c_5 = 2 \mid c_{1:4}) = \frac{2}{4+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 2,$ $m_2 = 2,$ $m_3 = 1,$ $m_4 = 0$



$$p(c_6 = 1 \mid c_{1:5}) = \frac{2}{5+1}$$

$$p(c_6 = 3 \mid c_{1:5}) = \frac{1}{5+1}$$

$$p(c_6 = 2 \mid c_{1:5}) = \frac{2}{5+1}$$

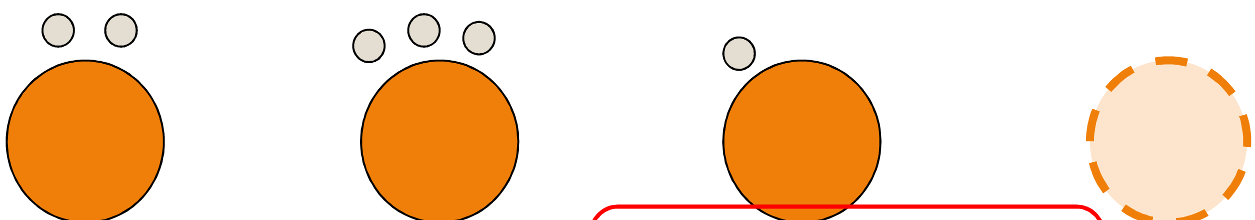
$$p(c_6 = 4 \mid c_{1:5}) = \frac{1}{5+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 2,$ $m_2 = 3,$ $m_3 = 1,$ $m_4 = 0$



$$p(c_7 = 1 \mid c_{1:6}) = \frac{2}{6+1}$$

$$p(c_7 = 3 \mid c_{1:6}) = \frac{1}{6+1}$$

$$p(c_7 = 2 \mid c_{1:6}) = \frac{3}{6+1}$$

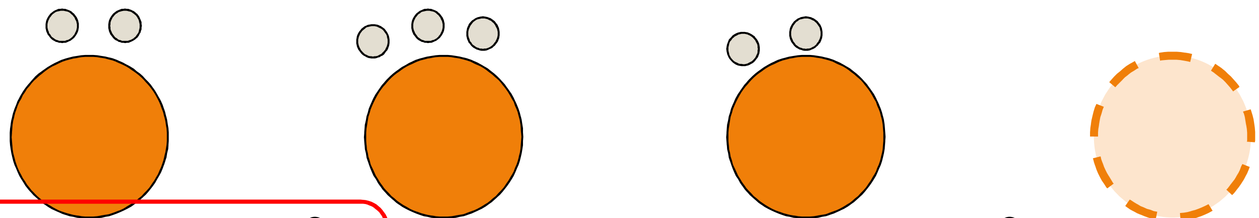
$$p(c_7 = 4 \mid c_{1:6}) = \frac{1}{6+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 2,$ $m_2 = 3,$ $m_3 = 2,$ $m_4 = 0$



$$p(c_8 = 1 \mid c_{1:3}) = \frac{2}{7+1}$$

$$p(c_8 = 3 \mid c_{1:3}) = \frac{2}{7+1}$$

$$p(c_8 = 2 \mid c_{1:3}) = \frac{3}{7+1}$$

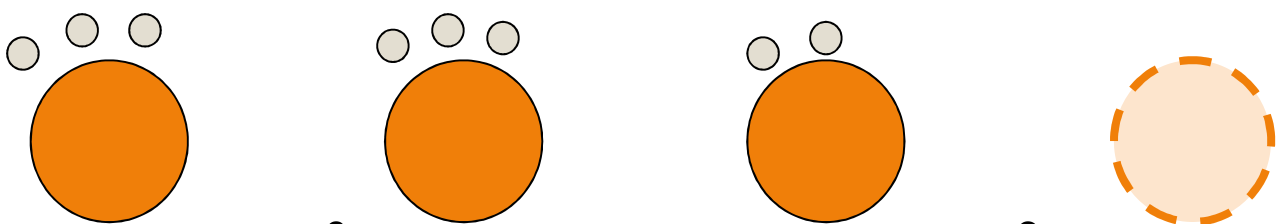
$$p(c_8 = 4 \mid c_{1:3}) = \frac{1}{7+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 3,$ $m_2 = 3,$ $m_3 = 2,$ $m_4 = 0$



$$p(c_9 = 1 \mid c_{1:3}) = \frac{3}{8+1}$$

$$p(c_9 = 3 \mid c_{1:3}) = \frac{2}{8+1}$$

$$p(c_9 = 2 \mid c_{1:3}) = \frac{3}{8+1}$$

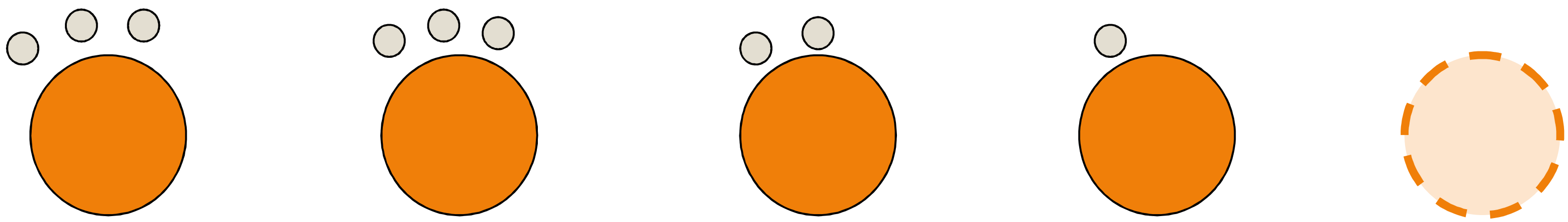
$$p(c_9 = 4 \mid c_{1:3}) = \frac{1}{8+1}$$

Chinese Restaurant Process (CRP)

$$p(c_i = k \mid c_1, \ldots, c_{i-1}) = \begin{cases} \dfrac{m_k}{i-1+\alpha} & k \le K_+ \\ \dfrac{\alpha}{i-1+\alpha} & k = K+1 \end{cases}$$

Parameter $\alpha = 1$

$m_1 = 3,$ $m_2 = 3,$ $m_3 = 2,$ $m_4 = 1$ $m_5 = 5$



$$p(c_{10} = 1 \mid c_{1:4}) = \frac{3}{9+1} \qquad p(c_{10} = 3 \mid c_{1:4}) = \frac{2}{9+1} \qquad p(c_{10} = 5 \mid c_{1:4}) = \frac{1}{9+1}$$

$$p(c_{10} = 2 \mid c_{1:4}) = \frac{3}{9+1} \qquad p(c_{10} = 4 \mid c_{1:4}) = \frac{1}{9+1}$$

CRP: Gibbs sampling

- Gibbs sampler requires full conditional

$$p(c_i = k \mid \mathbf{c}_{-i}, \mathbf{X}) \propto p(\mathbf{X} \mid \mathbf{c}).p(c_i = k \mid \mathbf{c}_{-i})$$

- Finite Mixture Model:

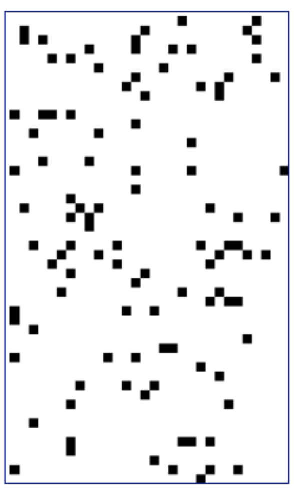
$$p(c_i = k \mid \mathbf{c}_{-i}) = \frac{m_{-i,k} + \frac{\alpha}{K}}{N - 1 + \alpha}$$

- Infinite Mixture Model:

$$p(c_i = k \mid \mathbf{c}_{-i}) = \begin{cases} \frac{m_{-i,k}}{N-1+\alpha} & m_{-i,k} > 0 \\ \frac{\alpha}{N-1+\alpha} & k = K_{-i} + 1 \\ 0 & \text{otherwise} \end{cases}$$

Beyond the limit of single label

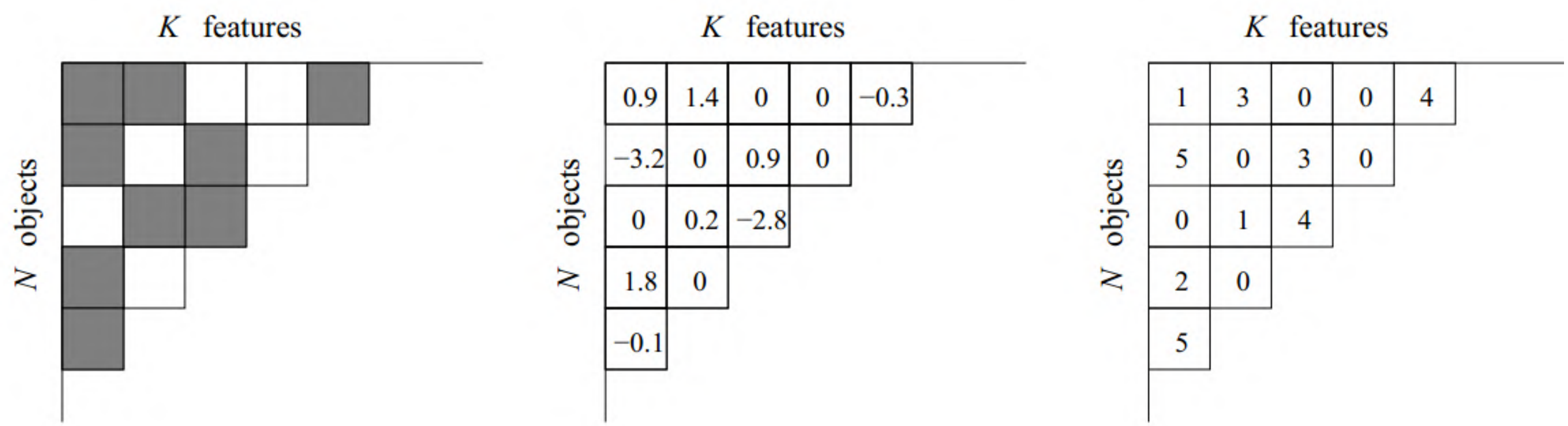
- In Latent Class Models:
  - Each object (word) has only one latent label (topic)
  - Finite number of latent labels: LDA
  - Infinite number of latent labels: DPM
- In Latent Feature (latent structure) Models:
  - Each object (graph) has multiple latent features (entities)
  - Finite number of latent features: Finite Feature Model (FFM)
  - Infinite number of latent features: Indian Buffet Process (IBP)



- Rows are data points
- Columns are latent features

- Movie Preference Example:
  - Rows are movies: *Rise of the Planet of the Apes*
  - Columns are latent features:
    - Made in U.S.
    - Is Science fiction
    - Has apes in it …

Latent Feature Model



$\boldsymbol{Z}$ continuous $\boldsymbol{F}$ discrete $\boldsymbol{F}$

- $\boldsymbol{F}$: latent feature matrix
- $\boldsymbol{Z}$: binary matrix
- $\boldsymbol{V}$: value matrix

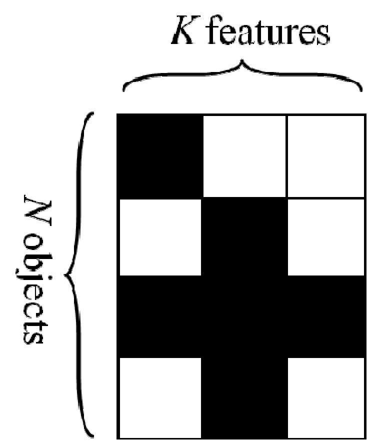
$$\mathbf{F} = \mathbf{Z} \otimes \mathbf{V}$$

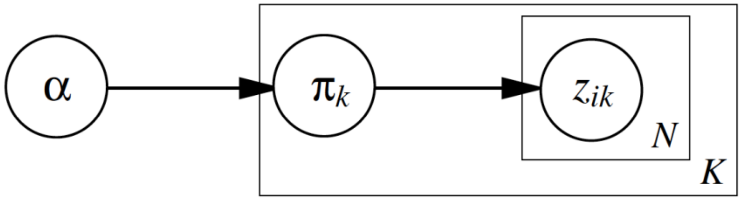
- With p($\boldsymbol{F}$)=p($\boldsymbol{Z}$). p($\boldsymbol{V}$)

Finite Feature Model

- Generating ***Z*** : (N*K) binary matrix
  - For each column k, draw π_k from beta distribution
  - For each object, flip a coin by z_{ik}

$$\begin{cases} \pi_k \mid \alpha \sim \text{Beta}(\frac{\alpha}{K},1) \\ z_{ik} \mid \pi_k \sim \text{Bernoulli}(\pi_k) \end{cases} \Leftrightarrow (\pi_k, 1-\pi_k) \sim Dir(\frac{\alpha}{K},1)$$





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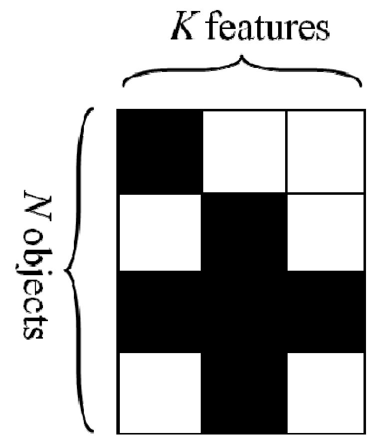
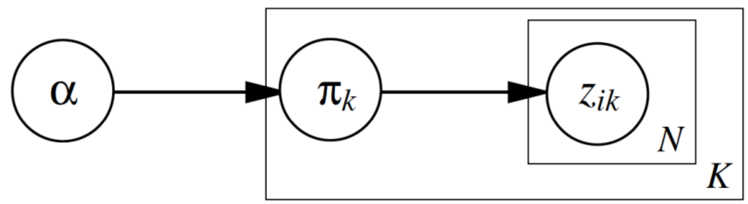


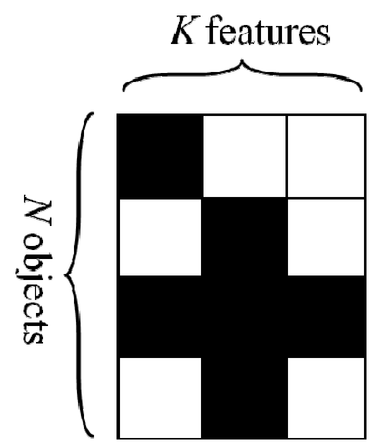


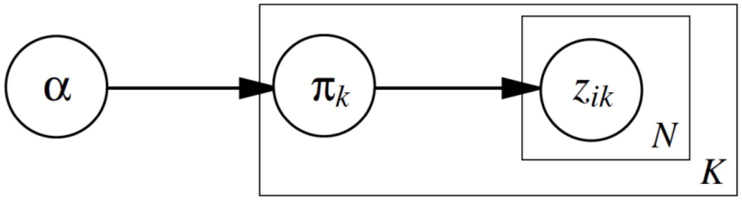
- Distribution of Z :

Finite Feature Model

- Generating ***Z*** : (N*K) binary matrix
  - For each column k, draw π_k from beta distribution
  - For each object, flip a coin by z_{ik}

$$\begin{cases} \pi_k \mid \alpha \sim \text{Beta}(\frac{\alpha}{K}, 1) \\ z_{ik} \mid \pi_k \sim \text{Bernoulli}(\pi_k) \end{cases} \Longleftrightarrow (\pi_k, 1-\pi_k) \sim Dir(\frac{\alpha}{K}, 1)$$

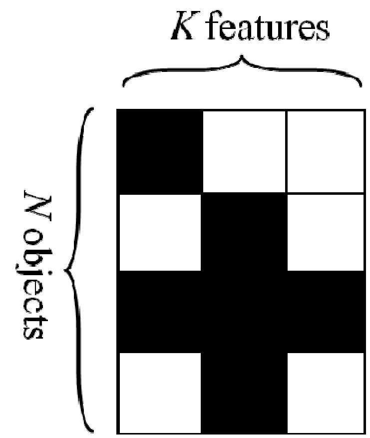




- Distribution of Z :

$$p(\mathbf{Z} \mid \boldsymbol{\pi}) = \prod_{k=1}^{K} \prod_{i=1}^{N} p(z_{ik} \mid \pi_k) = \prod_{k=1}^{K} \pi_k^{m_k} (1-\pi_k)^{N-m_k}$$

Finite Feature Model



- Generating ***Z*** : (N*K) binary matrix
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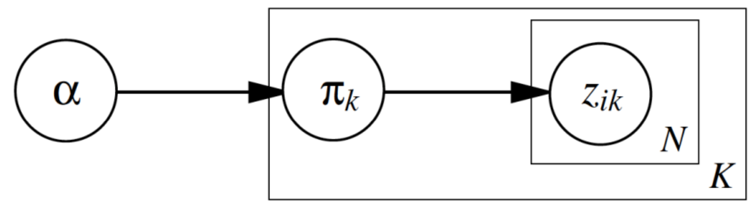

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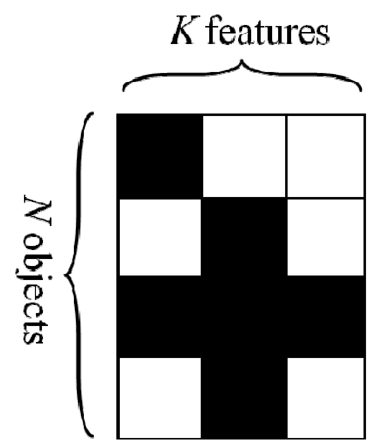

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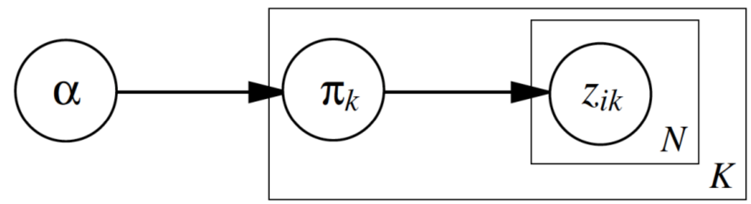

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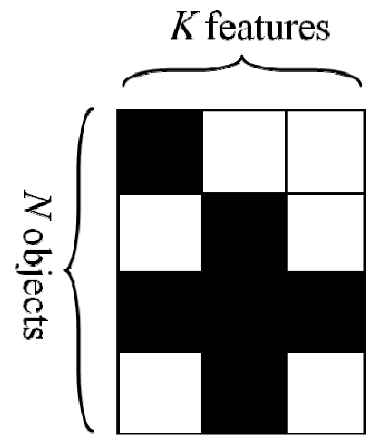

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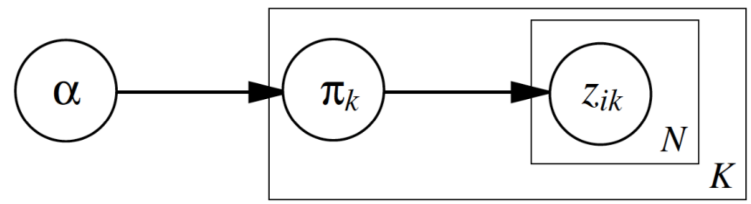

- ***Z*** is sparse:

$$\mathbb{E}\left[1^T \mathbf{Z} 1\right] = K\mathbb{E}\left[1^T \mathbf{Z}\right] = K\sum_{i=1}^{N}\mathbb{E}\left[z_{ik}\right] = KN\mathbb{E}\left[\pi_k\right] = N\frac{\frac{\alpha}{K}}{1+\frac{\alpha}{K}} \leq N\alpha$$

  - Even $K \rightarrow \infty$

Indian Buffet Process

1st Representation: $K \to \infty$

- Difficulty:
  - P(Z) -> 0
  - Solution: define equivalence classes on random binary feature matrices.
- *left-ordered form* function of binary matrices, *lof*(**Z**):
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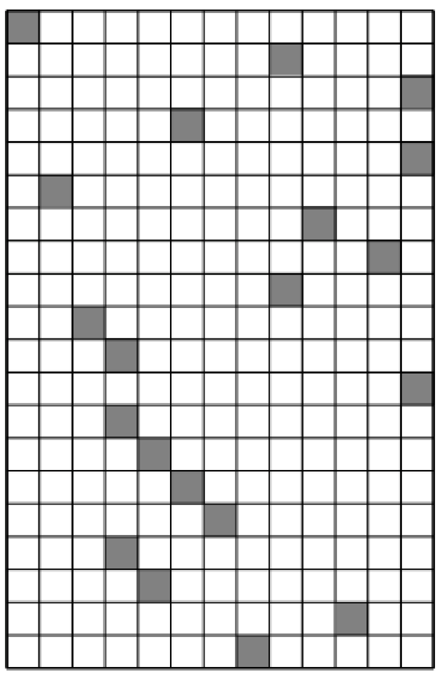

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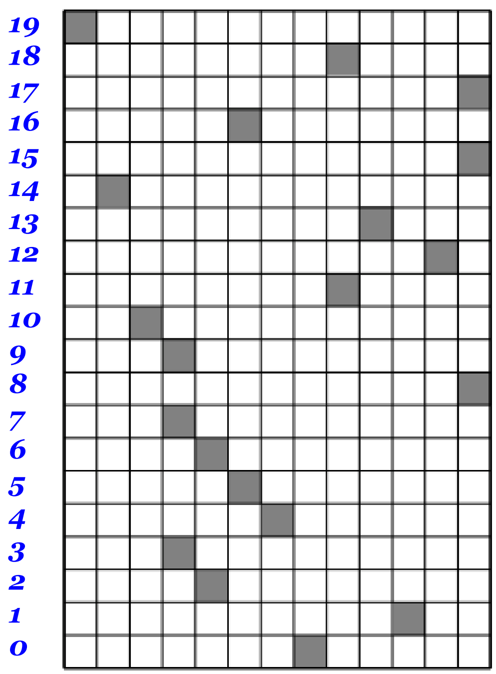

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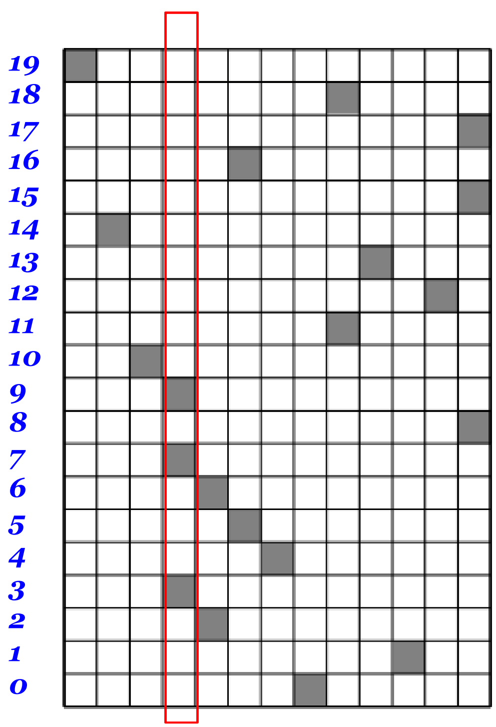

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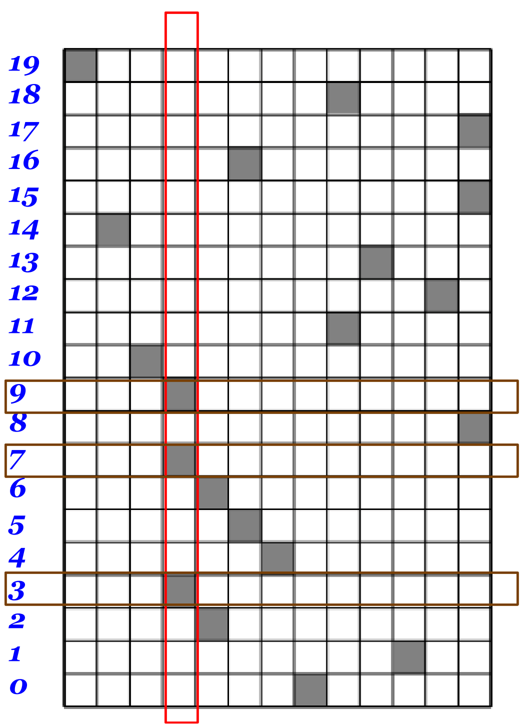

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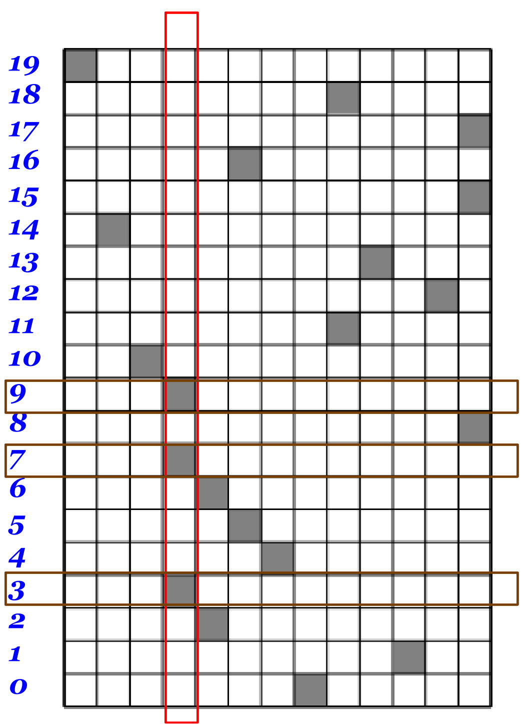

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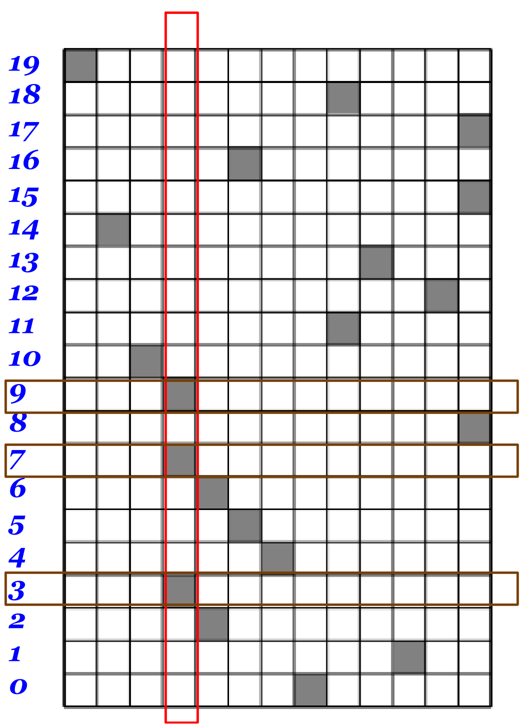
- Order features by h decreasingly



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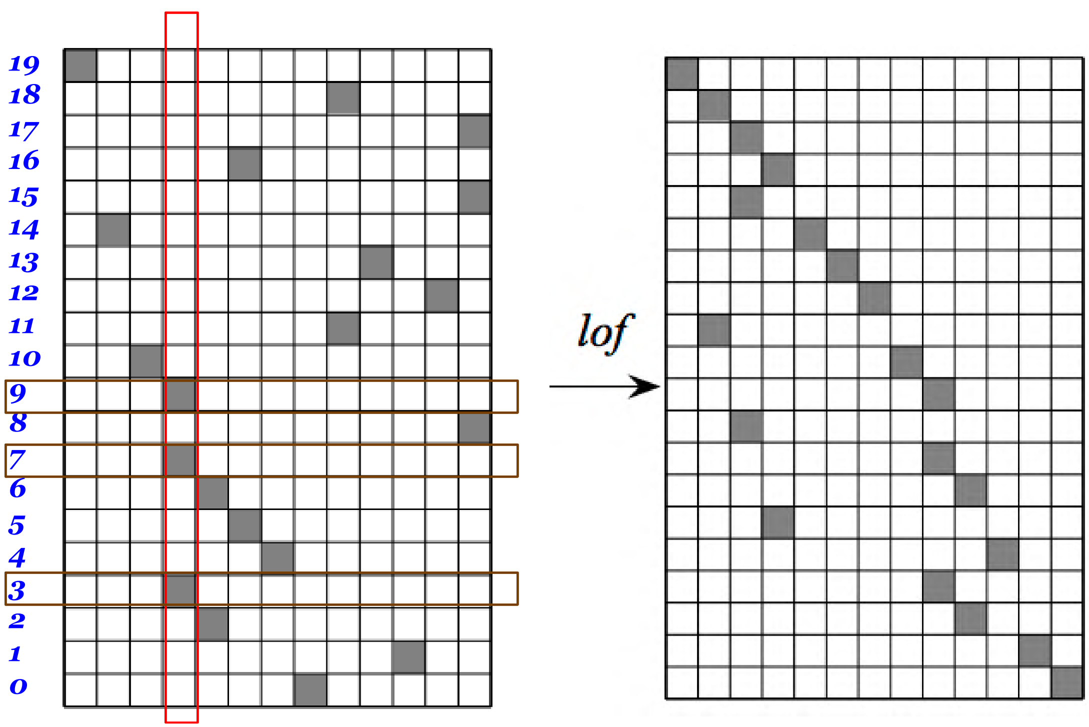

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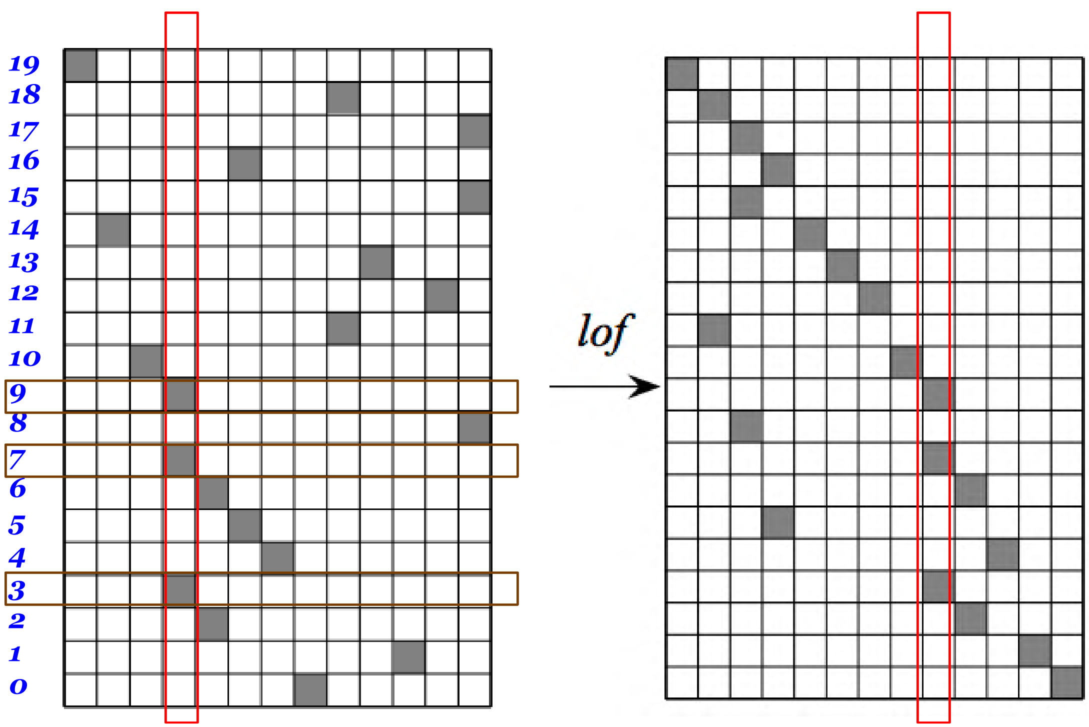

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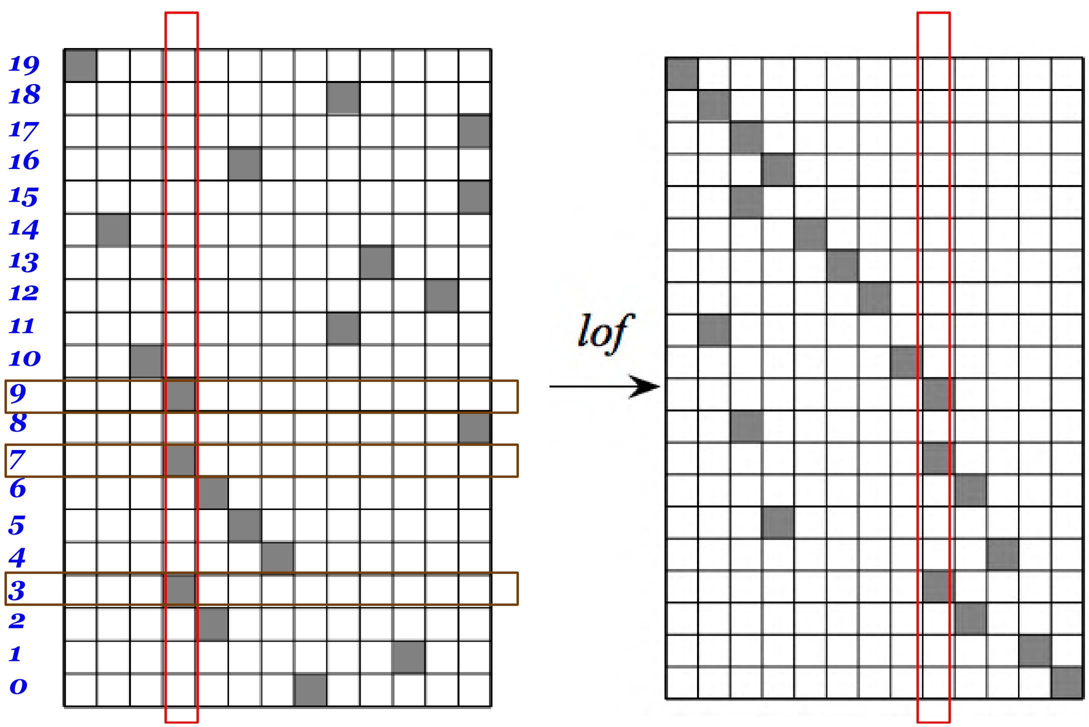
iff *lof*(**Z1**)=*lof*(**Z2**) = [**Z**]



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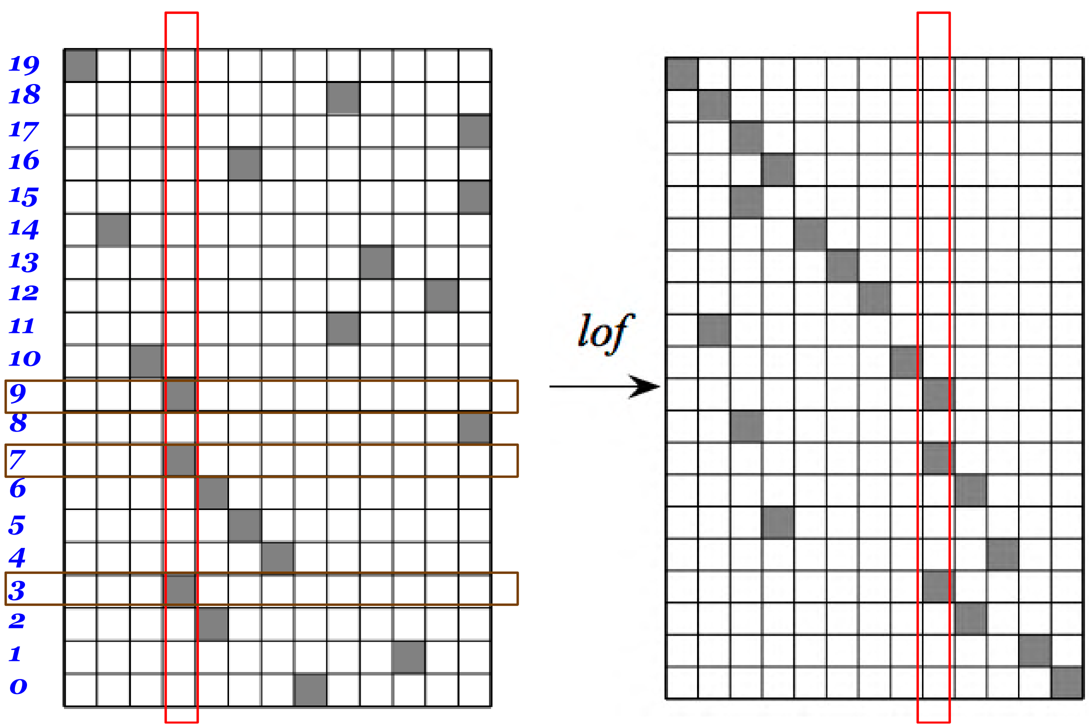

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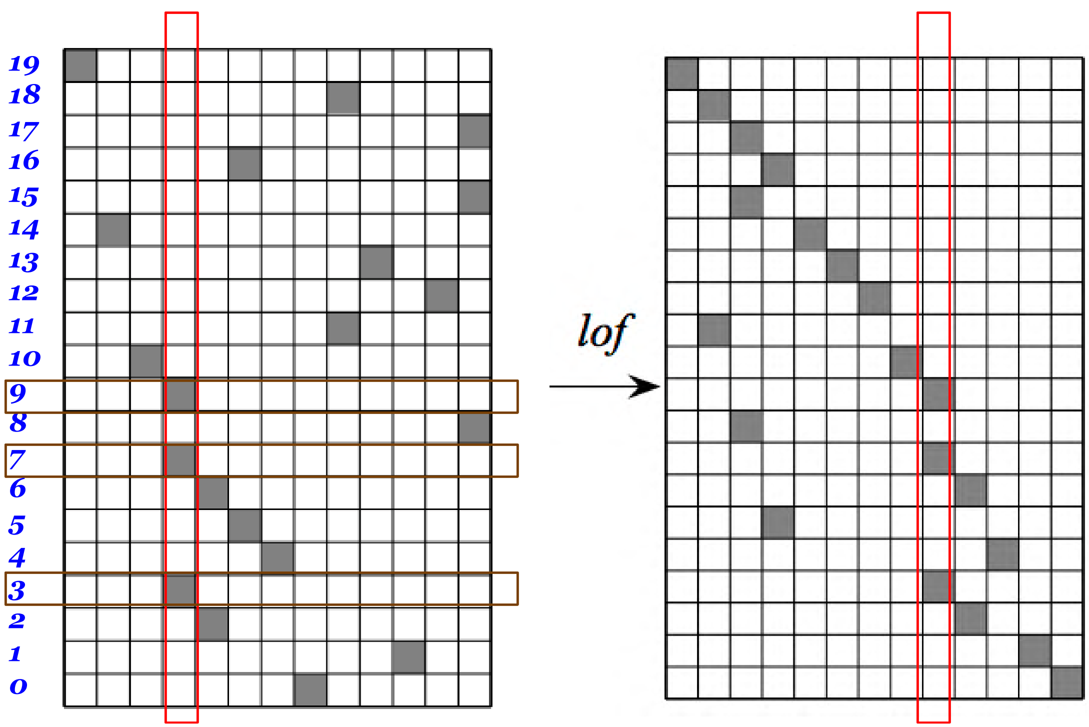

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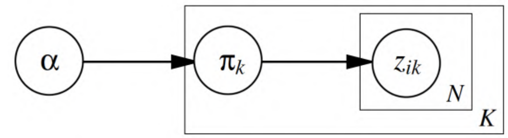

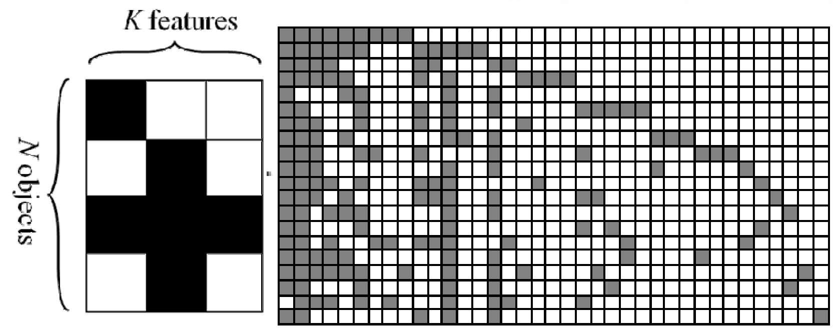

Given:

$$\Pr(\mathbf{Z} \mid \alpha) = \prod_{k=1}^{K} \frac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})}$$

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Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

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Indian Buffet Process
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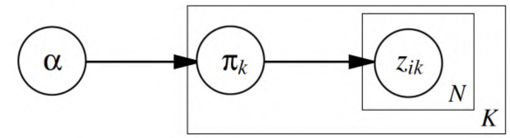

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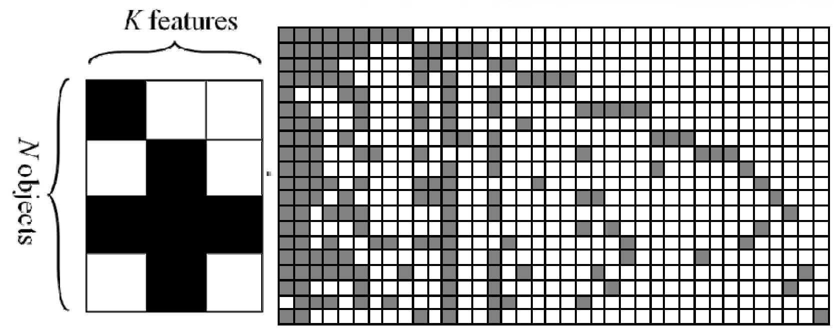

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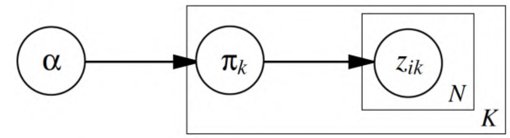

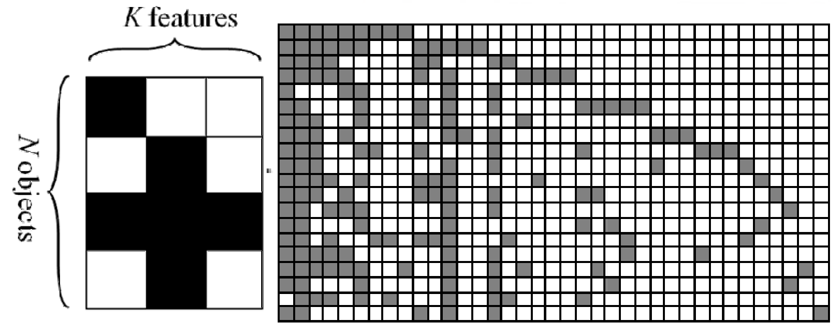

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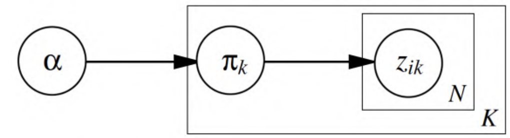

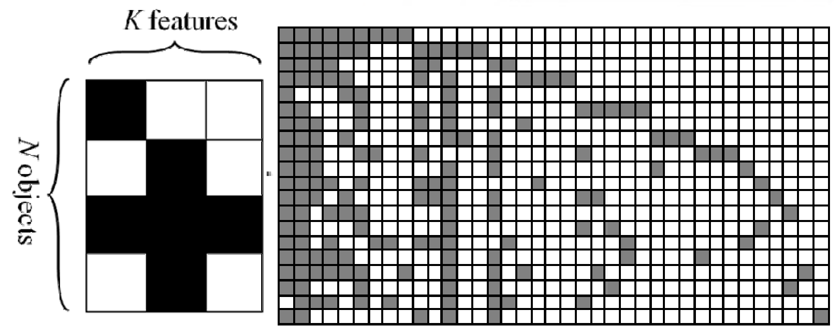

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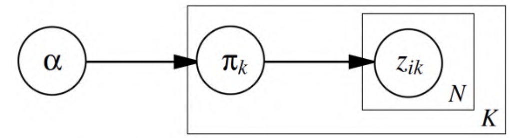

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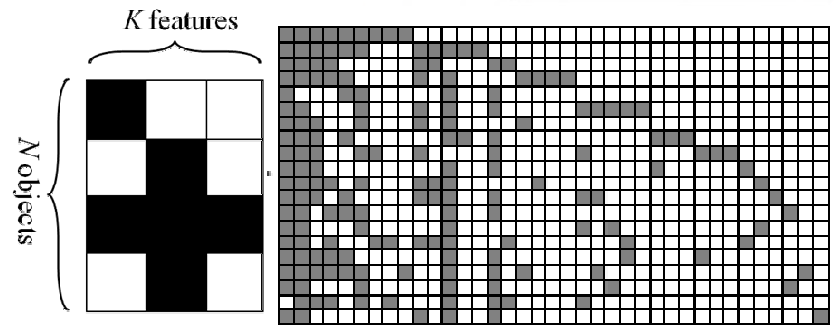

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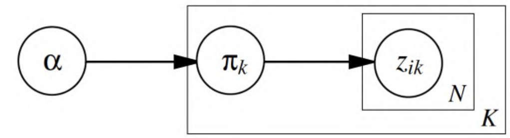

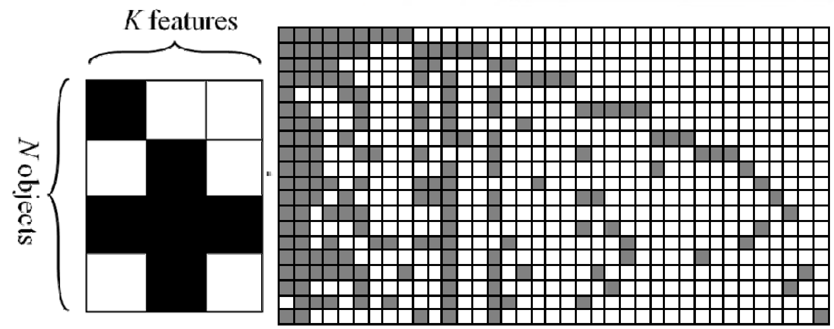

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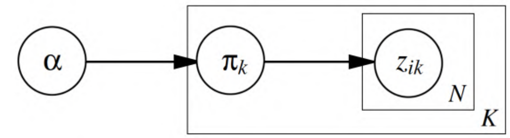

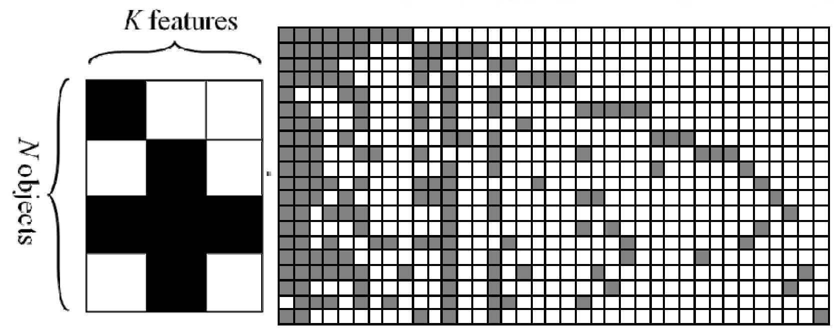

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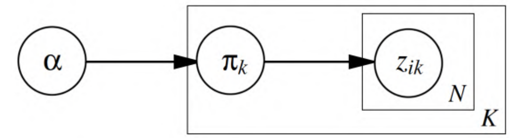
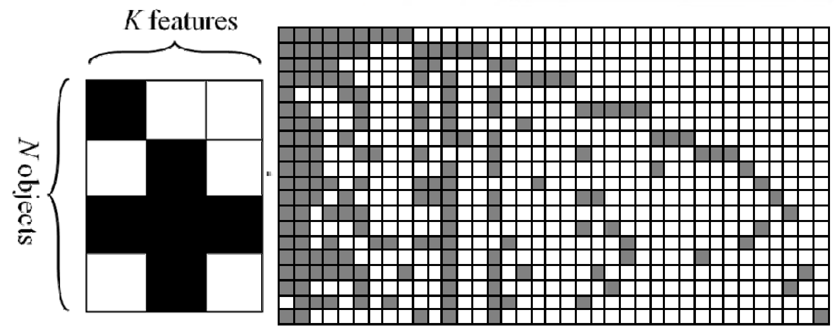
$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^K \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$\prod_{j=0}^{m_k-1} (j + \frac{\alpha}{K})$$

Indian Buffet Process
1st Representation: $K \to \infty$



Given:



$$\Pr(\mathbf{Z}\mid\alpha)=\prod_{k=1}^{K}\frac{\frac{\alpha}{K}\Gamma(m_k+\frac{\alpha}{K})\Gamma(N-m_k+1)}{\Gamma(N+1+\frac{\alpha}{K})}$$

$$card([Z])=\binom{K}{K_0\ldots K_{2^N-1}}=\frac{K!}{\prod_{h=0}^{2^N-1}K_h!}$$

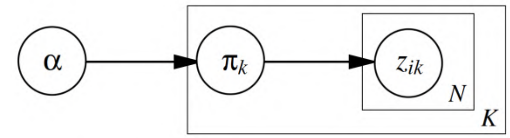
Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

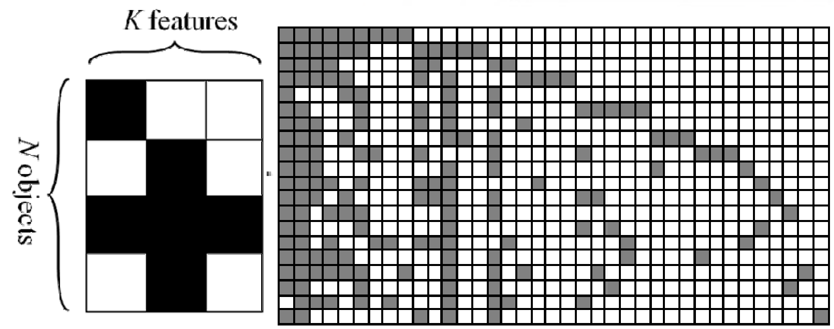
$$\Pr([Z]\mid\alpha)=\Pr(Z\mid\alpha)\cdot card([Z])$$

$$=\frac{K!}{\prod_{h=0}^{2^N-1}K_h!}\left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})}\right)^{K}\prod_{k=1}^{K_+}\frac{\Gamma(m_k+\frac{\alpha}{K})\Gamma(N-m_k+1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

Indian Buffet Process

1st Representation: $K \rightarrow \infty$





Given:

$$\Pr(\mathbf{Z} \mid \alpha) = \prod_{k=1}^{K} \frac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})}$$

$$card([Z]) = \binom{K}{K_0 \ldots K_{2^N-1}} = \frac{K!}{\prod_{h=0}^{2^N-1} K_h!}$$

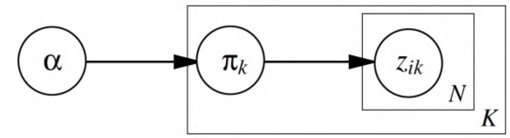
Derive when $K \rightarrow \infty$: ($K_+ < \infty$ almost surely)

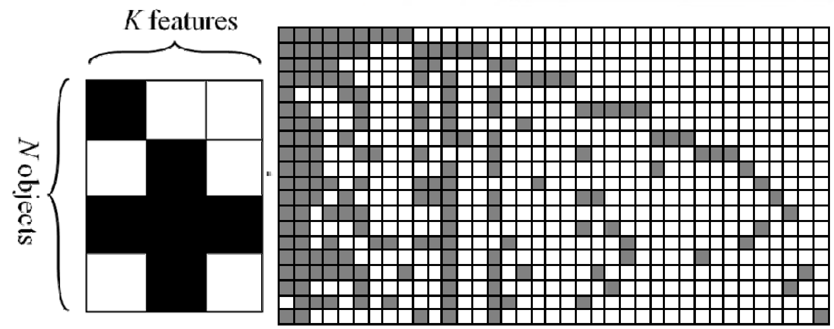
$$\Pr([Z] \mid \alpha) = \Pr(Z \mid \alpha) \cdot card([Z])$$

$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^K \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$(m_k - 1)! \cdot \frac{\alpha}{K}$$

Indian Buffet Process
1st Representation: $K \to \infty$





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Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

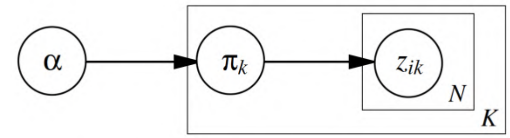
$$\Pr([Z] \mid \alpha) = \Pr(Z \mid \alpha) \cdot card([Z])$$

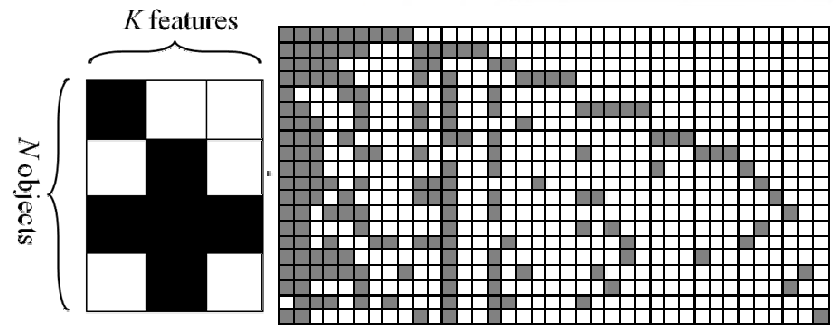
$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^{K} \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$(m_k - 1)! \cdot \frac{\alpha}{K} \qquad \frac{(N - m_k)!}{N!}$$

Indian Buffet Process

1^{st} Representation: $K \to \infty$





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Derive when $K \to \infty$: $(K_+ < \infty$ almost surely)

$$\Pr([Z]\mid\alpha) = \Pr(Z\mid\alpha)\cdot card([Z])$$

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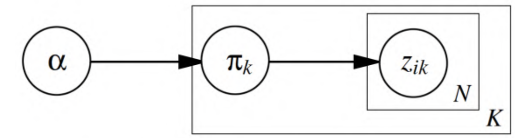
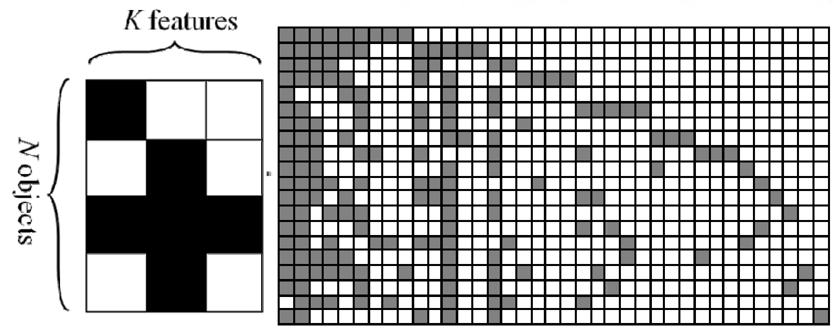
$$\prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

Indian Buffet Process

1st Representation: $K \to \infty$





Given:

$$\begin{cases} \Pr(\mathbf{Z}|\alpha) = \prod_{k=1}^{K} \dfrac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N+1+\frac{\alpha}{K})} \\ card([Z]) = \binom{K}{K_0 \ldots K_{2^N-1}} = \dfrac{K!}{\prod_{h=0}^{2^N-1} K_h!} \end{cases}$$

Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

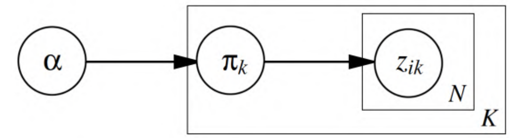
$$\Pr([Z]|\alpha) = \Pr(Z|\alpha) \cdot card([Z])$$

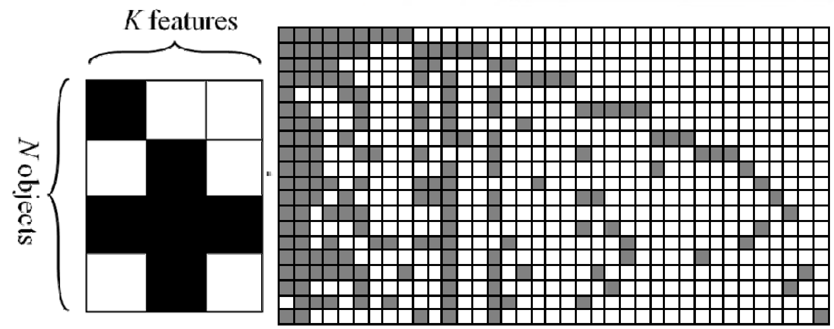
$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^K \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$\left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^K$$

$$\prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

Indian Buffet Process
1st Representation: $K \to \infty$





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Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

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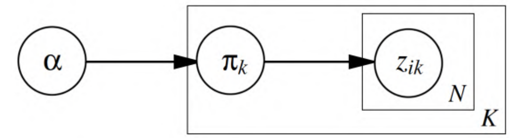
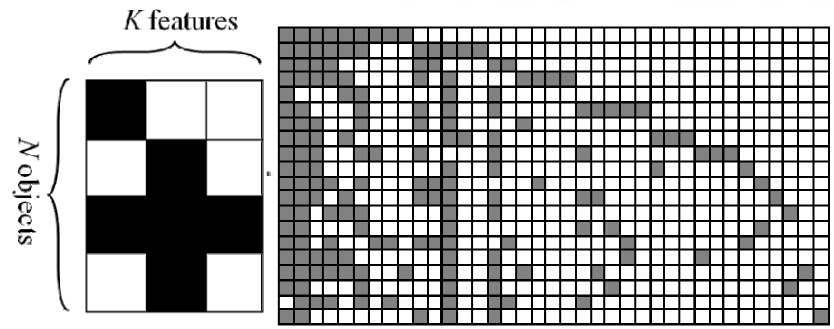
$$=\frac{K!}{\prod_{h=0}^{2^N-1}K_h!}\left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})}\right)^{K}\prod_{k=1}^{K_+}\frac{\Gamma(m_k+\frac{\alpha}{K})\Gamma(N-m_k+1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$\prod_{k=1}^{K_+}(m_k-1)!\cdot\frac{\alpha}{K}\frac{(N-m_k)!}{N!}$$

Indian Buffet Process

1st Representation: $K \to \infty$





Given:

$$
\left\{
\begin{aligned}
&\Pr(\mathbf{Z}\mid\alpha) = \prod_{k=1}^{K} \frac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})} \\
&card([Z]) = \binom{K}{K_0 \ldots K_{2^N-1}} = \frac{K!}{\prod_{h=0}^{2^N-1} K_h!}
\end{aligned}
\right.
$$

Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

$$\Pr([Z]\mid\alpha) = \Pr(Z\mid\alpha)\cdot card([Z])$$

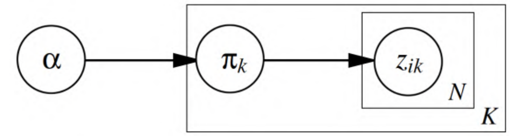
$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^{K} \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

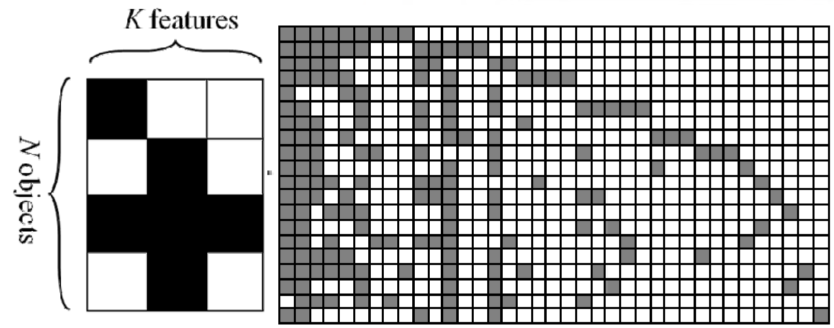
$$\left(\frac{\Gamma(\frac{\alpha}{K}+1)\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^{K}$$

$$\prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

Indian Buffet Process

1st Representation: $K \to \infty$





Given:

$$\begin{cases} \Pr(\mathbf{Z} \mid \alpha) = \prod_{k=1}^{K} \dfrac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})} \\ card([Z]) = \binom{K}{K_0 \ldots K_{2^N-1}} = \dfrac{K!}{\prod_{h=0}^{2^N-1} K_h!} \end{cases}$$

Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

$$\Pr([Z] \mid \alpha) = \Pr(Z \mid \alpha) \cdot card([Z])$$

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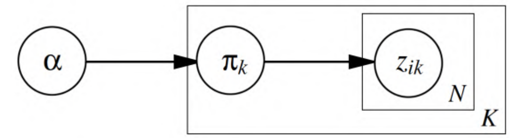
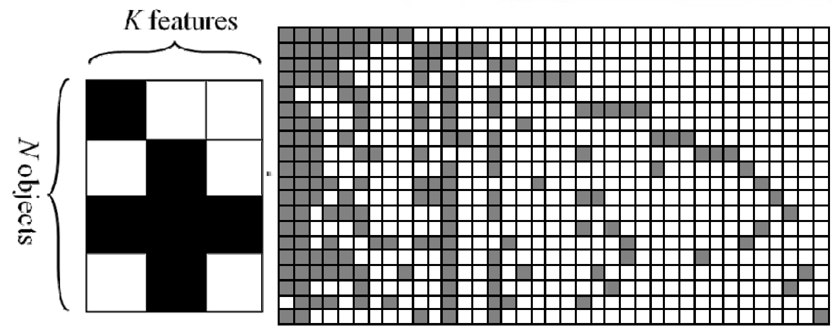
Indian Buffet Process

1st Representation: $K \to \infty$





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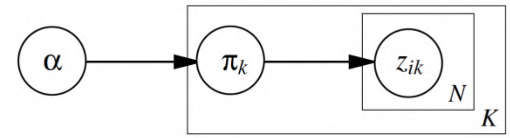
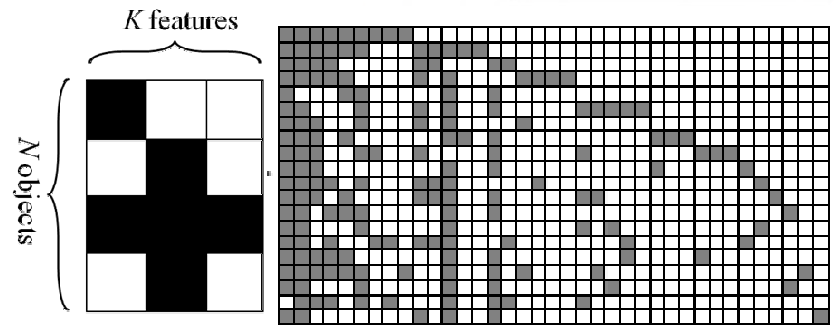
$$= \frac{K!}{\prod_{h=0}^{2^N-1} K_h!} \left(\frac{\frac{\alpha}{K}\Gamma(\frac{\alpha}{K})\Gamma(N+1)}{\Gamma(N+1+\frac{\alpha}{K})} \right)^K \prod_{k=1}^{K_+} \frac{\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(\frac{\alpha}{K})\Gamma(N+1)}$$

$$\left(\frac{N!}{\prod_{j=1}^{N}(j + \frac{\alpha}{K})} \right)^K$$

$$\prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

Indian Buffet Process
1st Representation: $K \to \infty$





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$$\Pr(\mathbf{Z}\mid\alpha)=\prod_{k=1}^{K}\frac{\frac{\alpha}{K}\Gamma(m_k+\frac{\alpha}{K})\Gamma(N-m_k+1)}{\Gamma(N+1+\frac{\alpha}{K})}$$

$$card([Z])=\binom{K}{K_0\ldots K_{2^N-1}}=\frac{K!}{\prod_{h=0}^{2^N-1}K_h!}$$

Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

$$\Pr([Z]\mid\alpha)=\Pr(Z\mid\alpha)\cdot card([Z])$$

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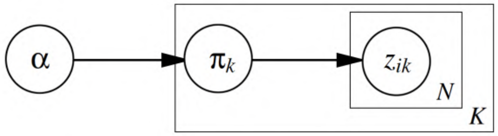
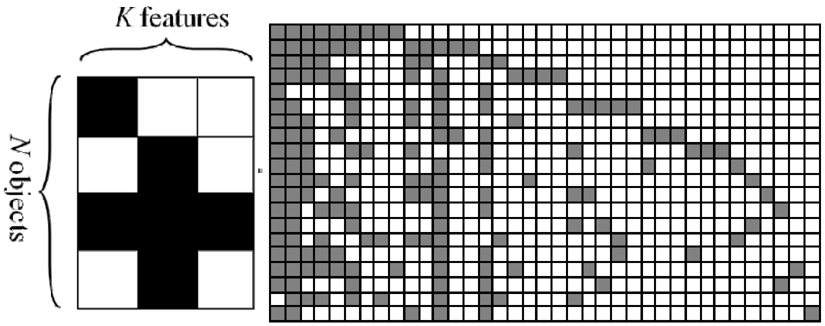
$$\prod_{k=1}^{K_+}(m_k-1)!\cdot\frac{\alpha}{K}\frac{(N-m_k)!}{N!}$$

Indian Buffet Process 1st Representation: $K \to \infty$





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$$card([Z]) = \binom{K}{K_0 \ldots K_{2^N-1}} = \frac{K!}{\prod_{h=0}^{2^N-1} K_h!}$$

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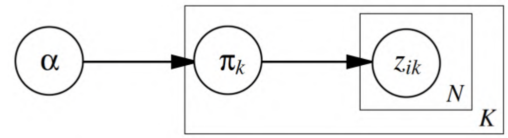
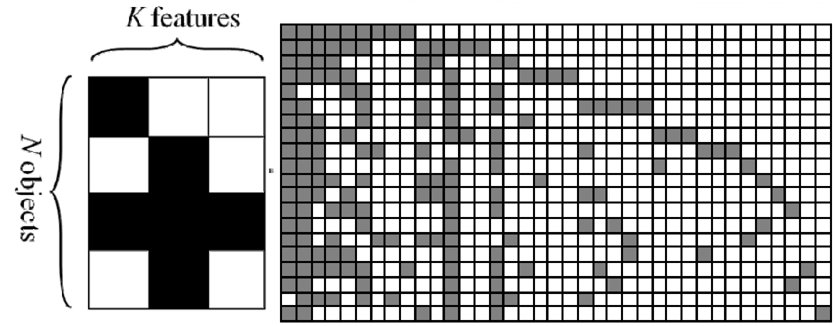
$$\left(\prod_{j=1}^{N} \frac{1}{(1 + \frac{\alpha / j}{K})} \right)^K \qquad \prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

Indian Buffet Process

1^{st} Representation: $K \to \infty$





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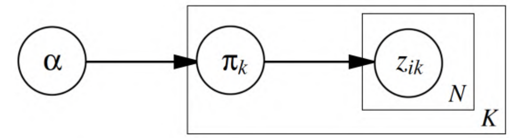
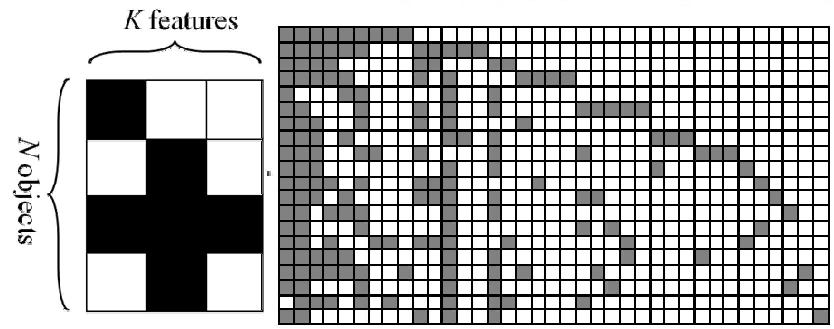
Indian Buffet Process

1st Representation: $K \to \infty$





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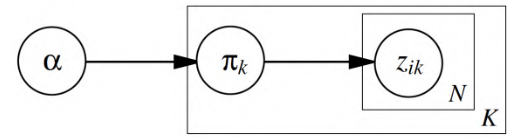
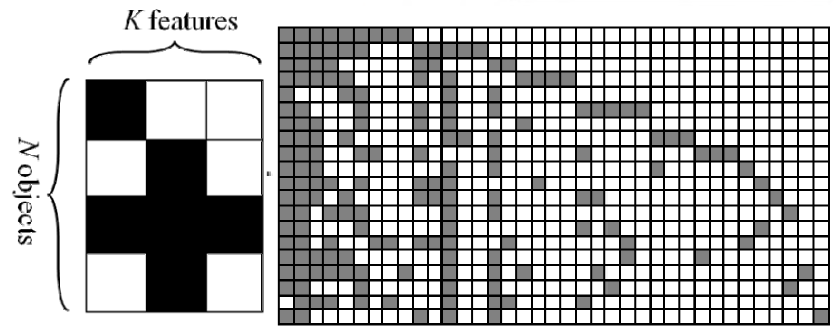
Indian Buffet Process

1st Representation: $K \to \infty$





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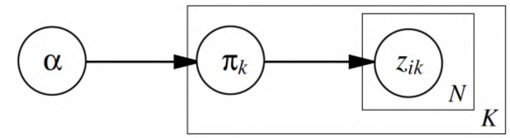
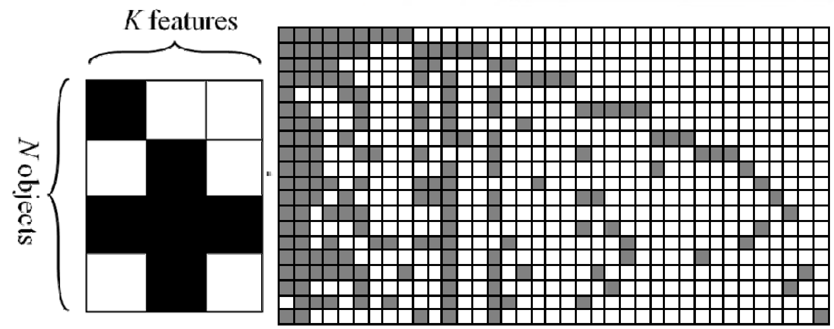
Indian Buffet Process
1st Representation: $K \to \infty$





Given:

$$\begin{cases} \Pr(\mathbf{Z} \mid \alpha) = \prod_{k=1}^{K} \dfrac{\frac{\alpha}{K}\Gamma(m_k + \frac{\alpha}{K})\Gamma(N - m_k + 1)}{\Gamma(N + 1 + \frac{\alpha}{K})} \\ card([Z]) = \begin{pmatrix} K \\ K_0 \ldots K_{2^N-1} \end{pmatrix} = \dfrac{K!}{\prod_{h=0}^{2^N-1} K_h!} \end{cases}$$

Derive when $K \to \infty$: ($K_+ < \infty$ almost surely)

$$\Pr([Z] \mid \alpha) = \Pr(Z \mid \alpha) \cdot card([Z])$$

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$$\prod_{j=1}^{N} \left(1 + \frac{\alpha / j}{K}\right)^{-K} \qquad \prod_{k=1}^{K_+} (m_k - 1)! \cdot \frac{\alpha}{K} \frac{(N - m_k)!}{N!}$$

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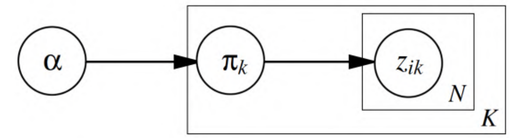

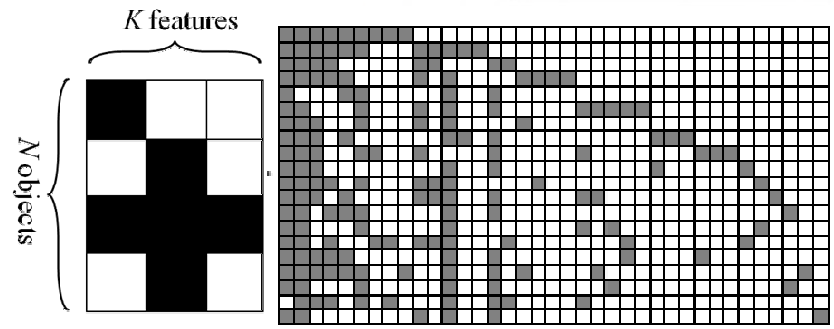

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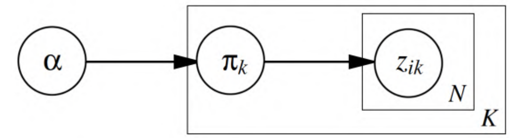

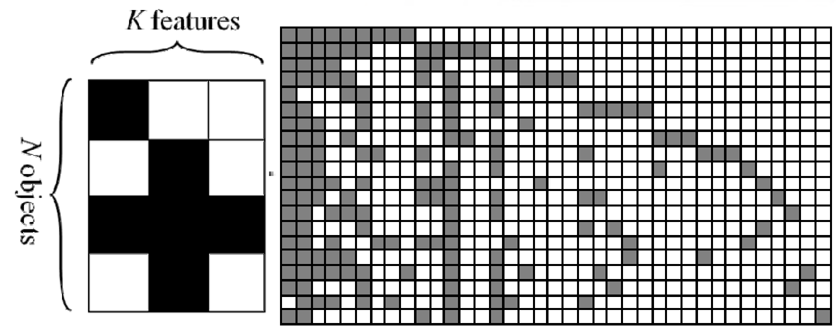

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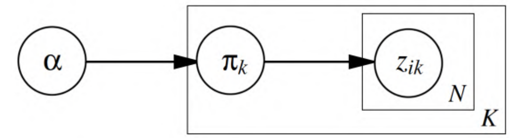

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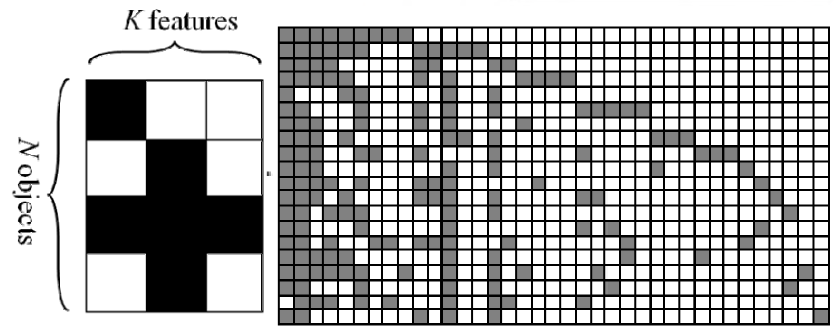

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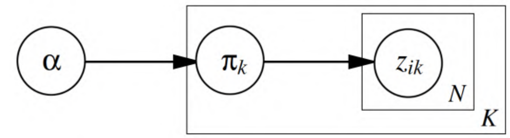

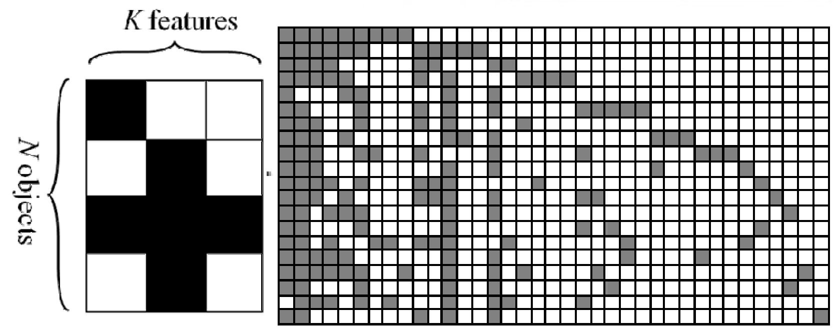

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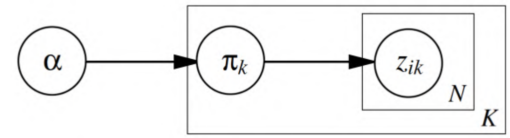

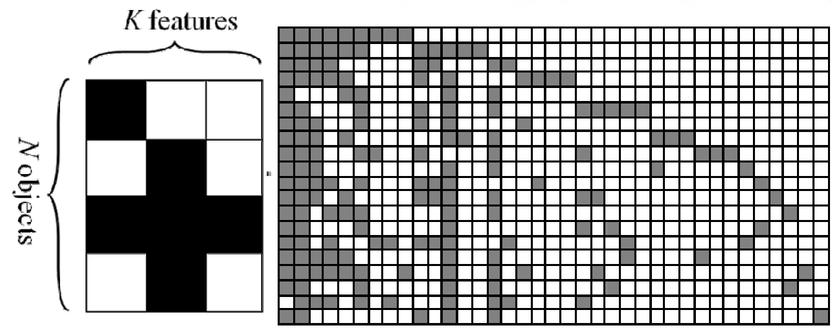

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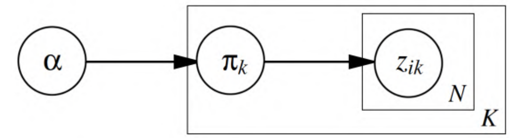

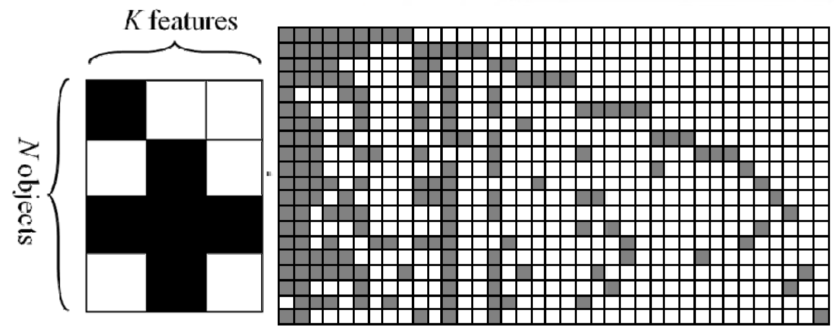

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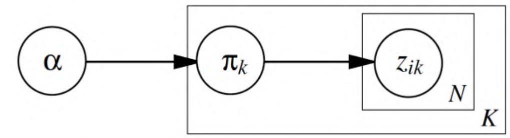

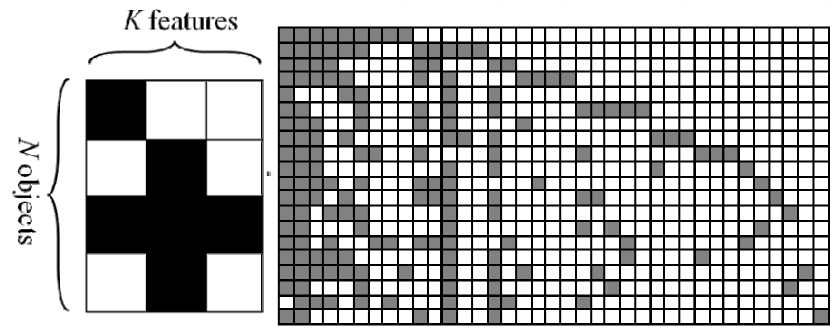

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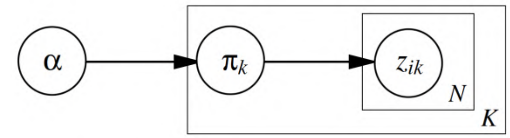

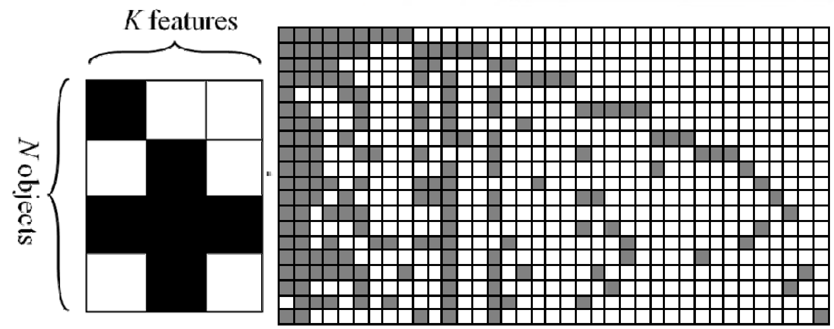

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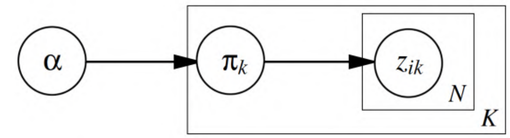

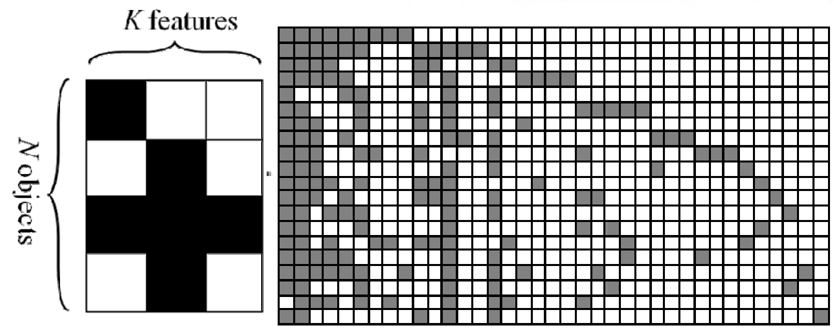

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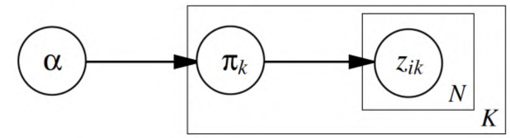

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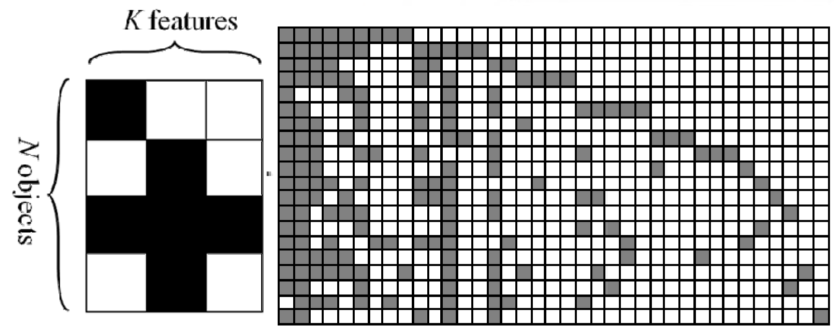

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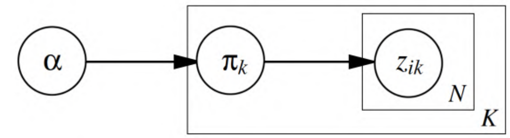

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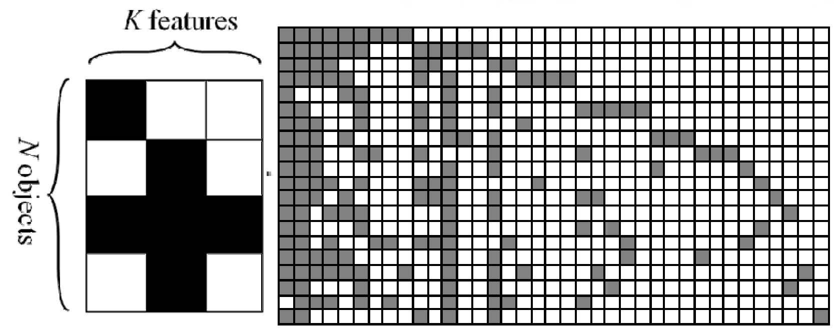

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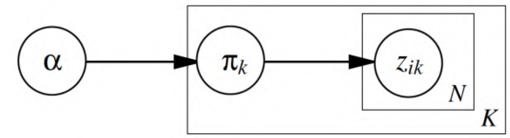

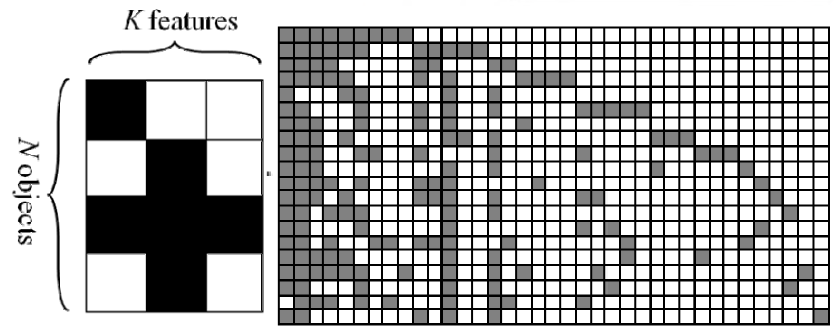

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$$= e^{-\alpha H_N} \quad \frac{\alpha^{K_+}}{\prod_{h=1}^{2^N-1} K_h!} \quad \prod_{k=1}^{K_+} (m_k - 1)! \frac{(N - m_k)!}{N!}$$

$$H_N = \sum_{j=1}^{N} \frac{1}{j}$$ Harmonic sequence sum

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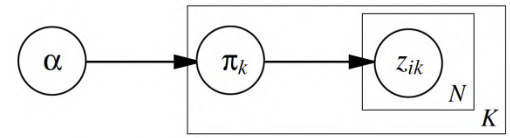

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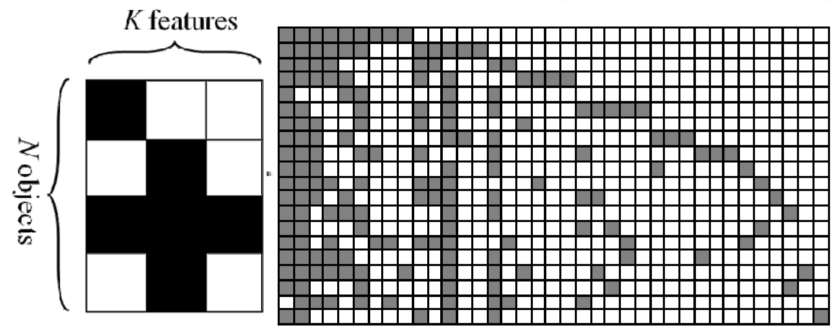

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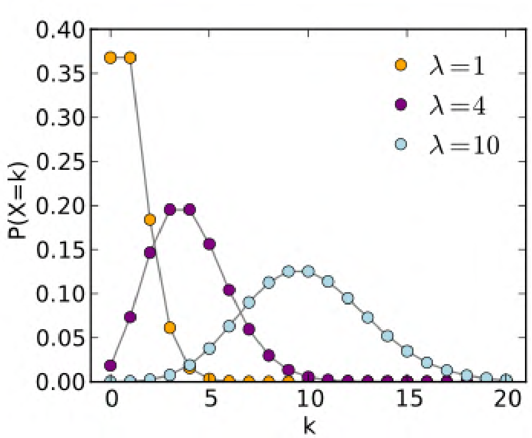
Note:

- $\Pr([Z] \mid \alpha)$ is well defined because $K_+ < \infty$ *a.s.*
- $\Pr([Z] \mid \alpha)$ depends on K_h :
  - The number of features (columns) with history h
  - Permute the rows (data points) does not change K_h (exchangeability)

Indian Buffet Process
2st Representation: Customers & Dishes

- Indian restaurant with infinitely many infinite dishes (columns)

Poission Distribution: $p(k \mid \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$



Indian Buffet Process
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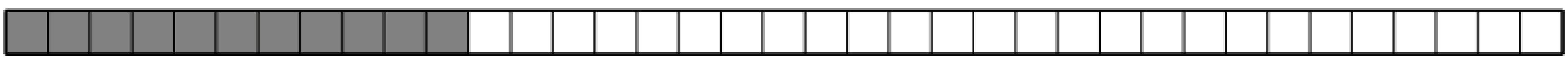
- Indian restaurant with infinitely many infinite dishes (columns)
  - The first customer tastes ***first*** $K_1^{(1)}$ dishes, sample $K_1^{(1)} \sim \text{Poission}(\alpha)$



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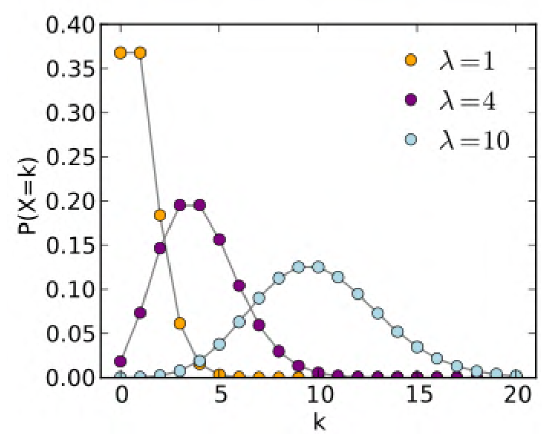

Indian Buffet Process
2st Representation: Customers & Dishes

- Indian restaurant with infinitely many infinite dishes (columns)
  - The first customer tastes ***first*** $K_1^{(1)}$ dishes, sample $K_1^{(1)} \sim \text{Poission}(\alpha)$
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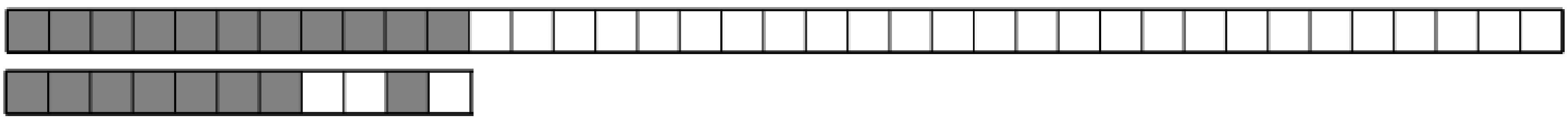

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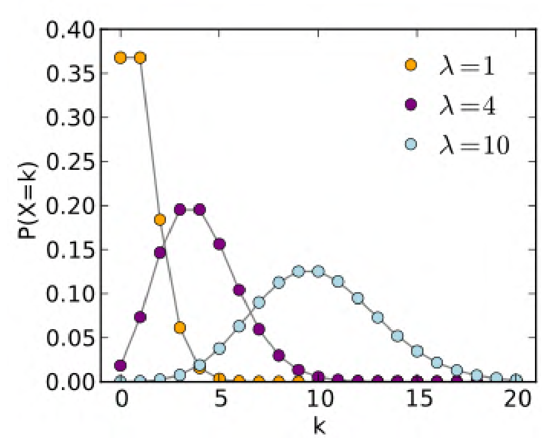

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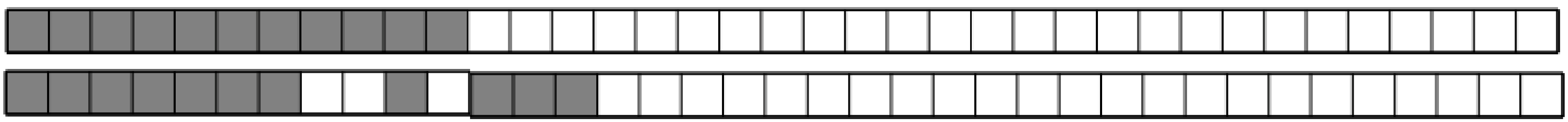

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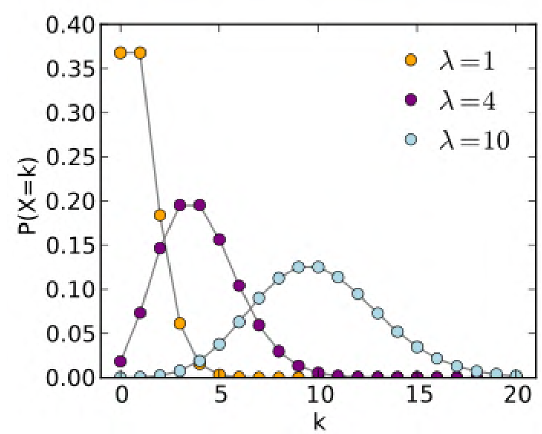

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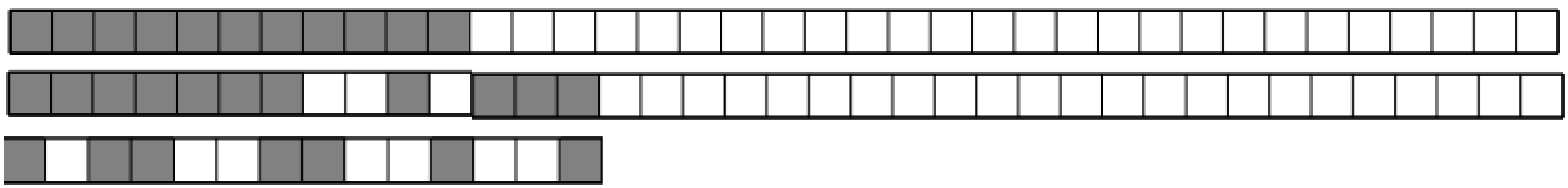

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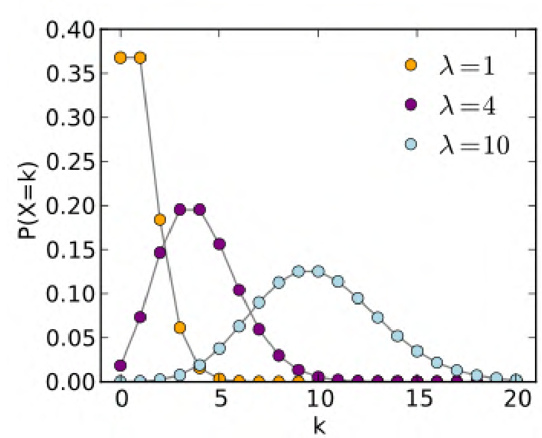

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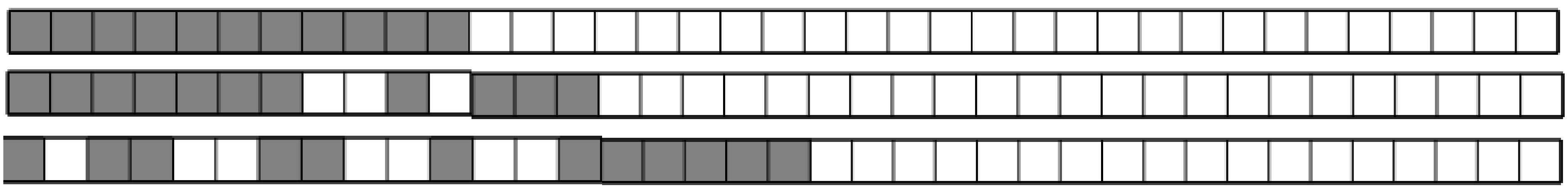

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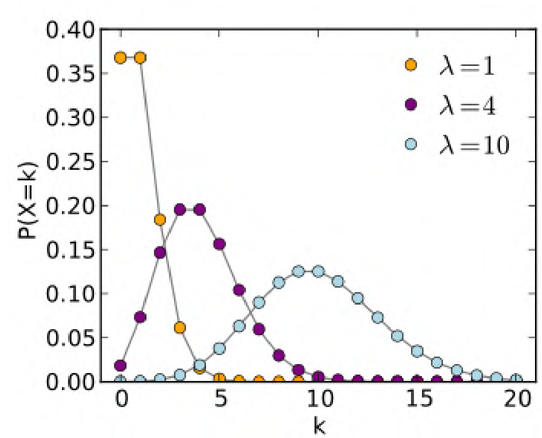

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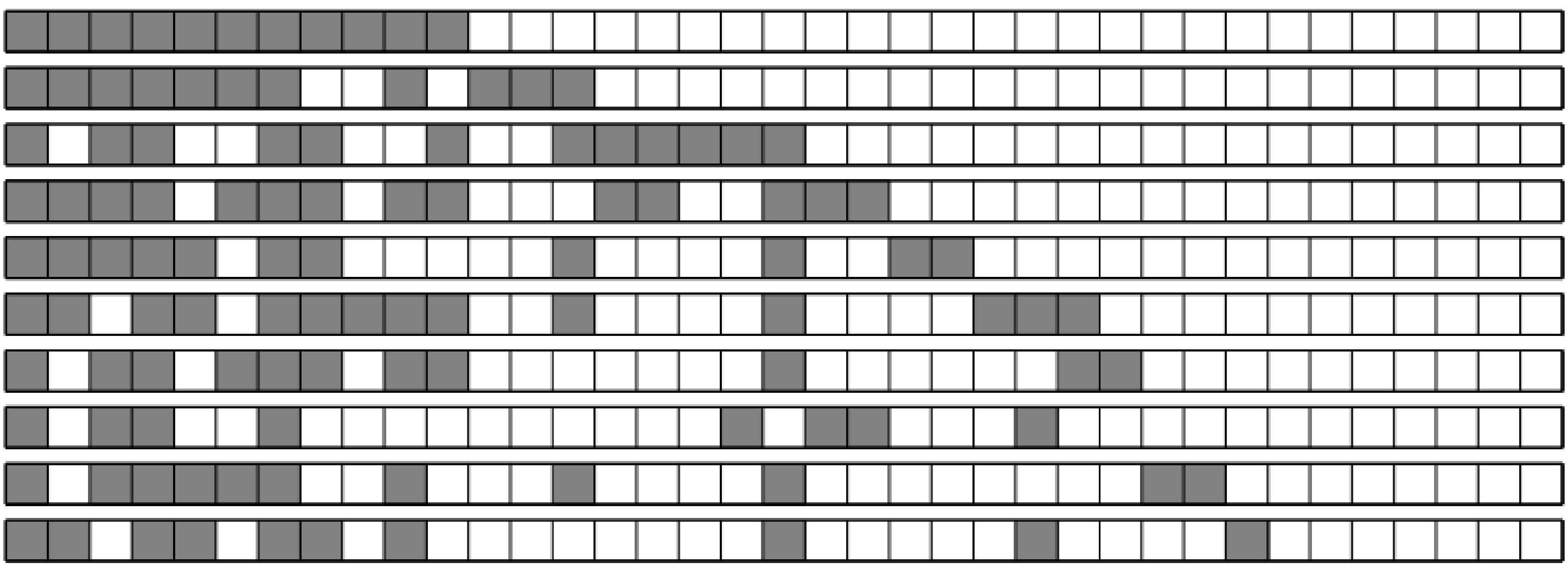

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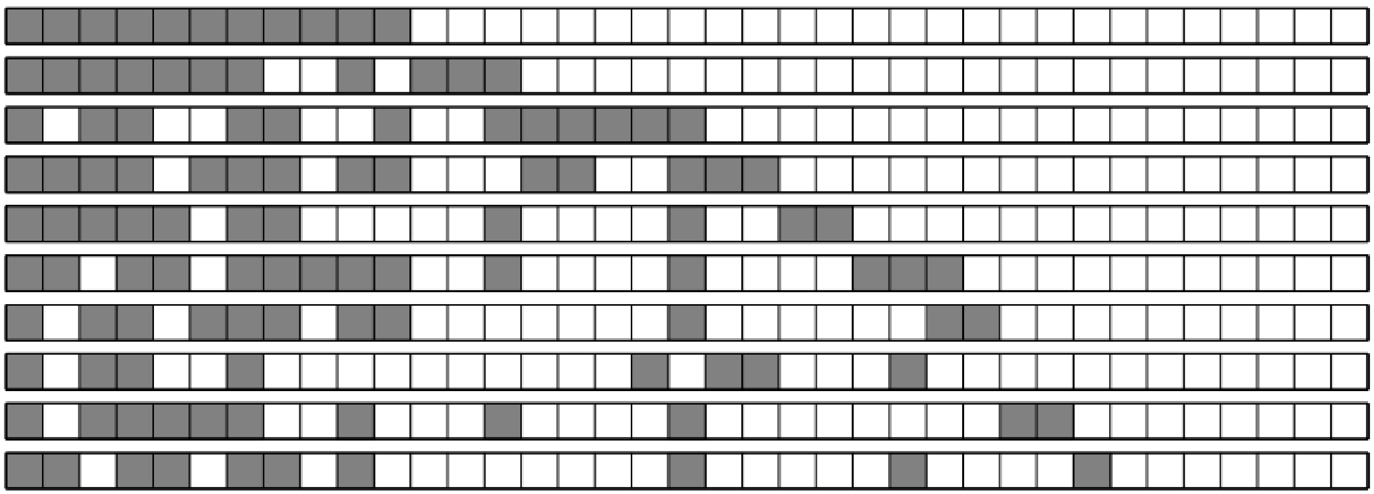

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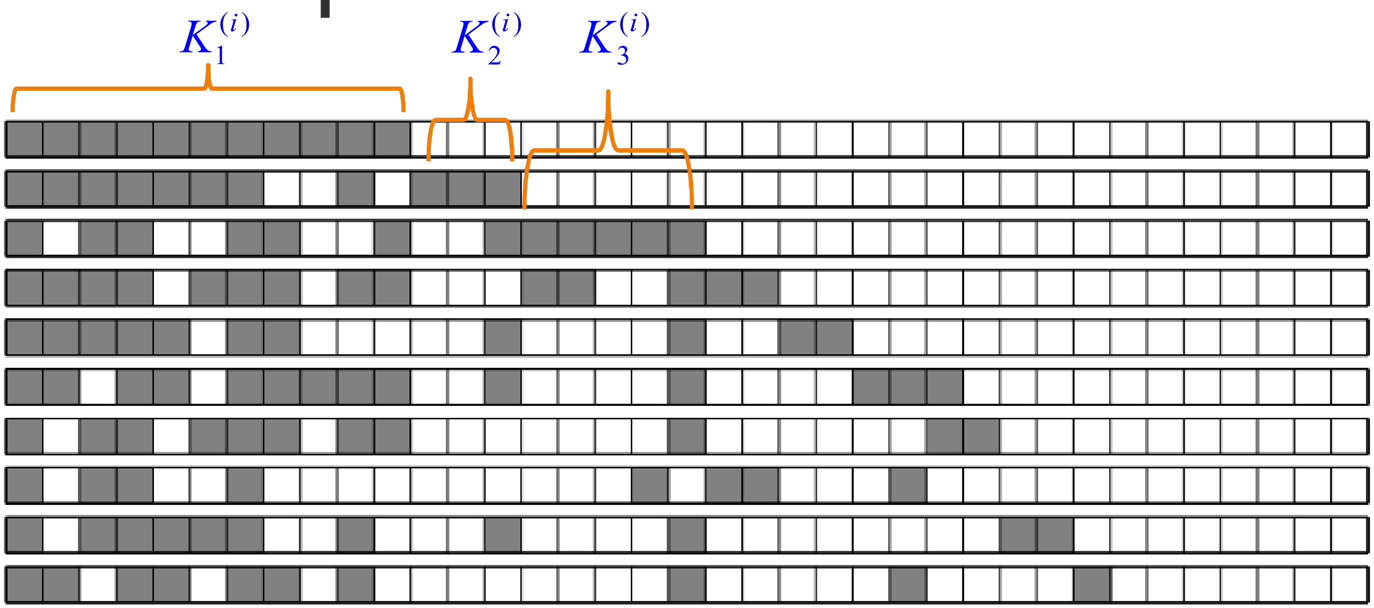

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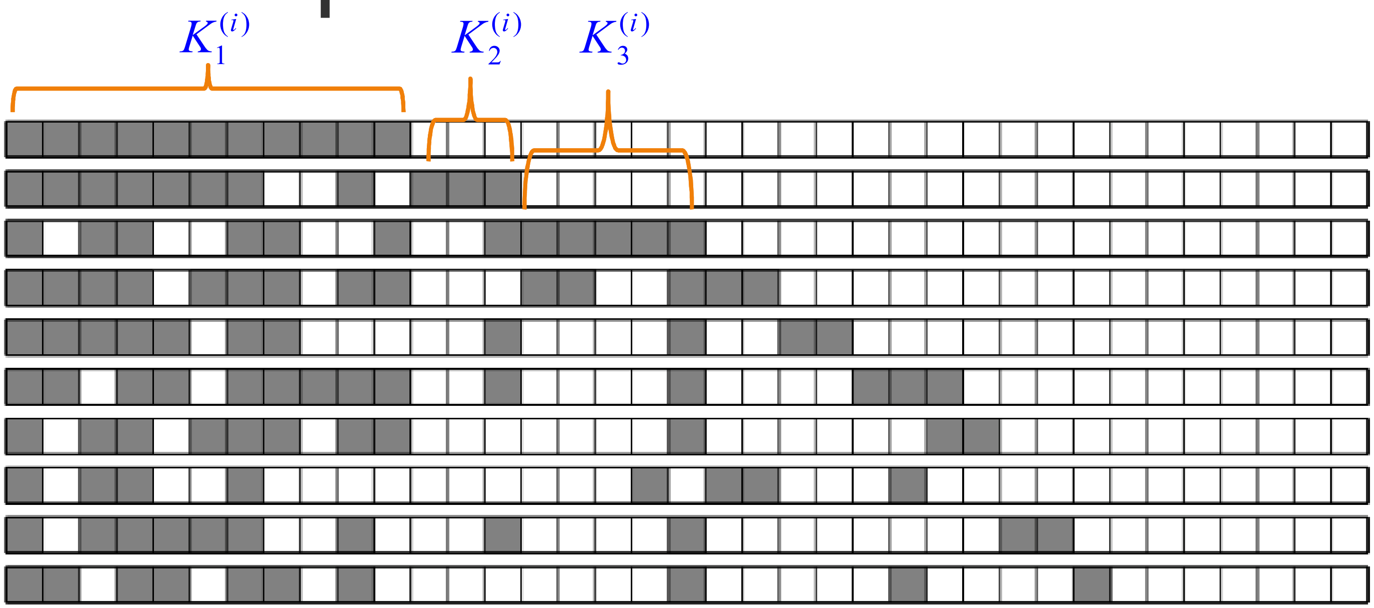

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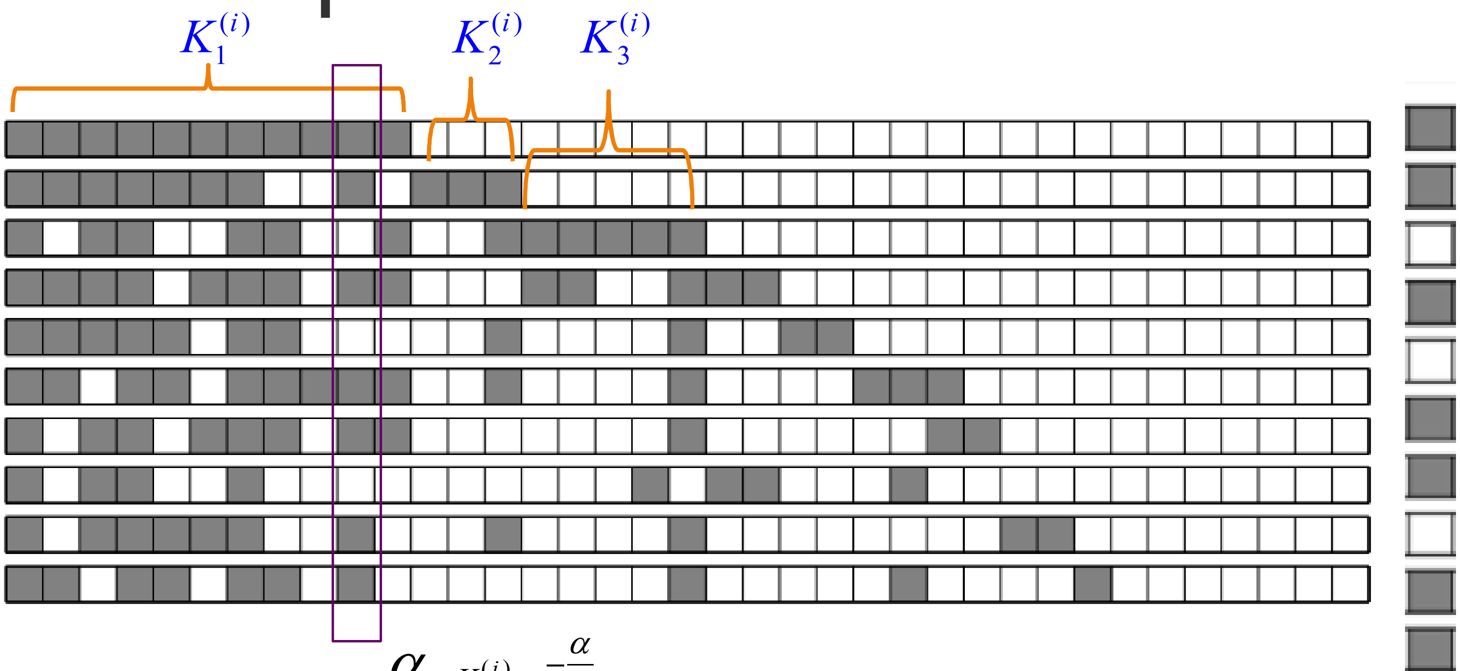

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Indian Buffet Process

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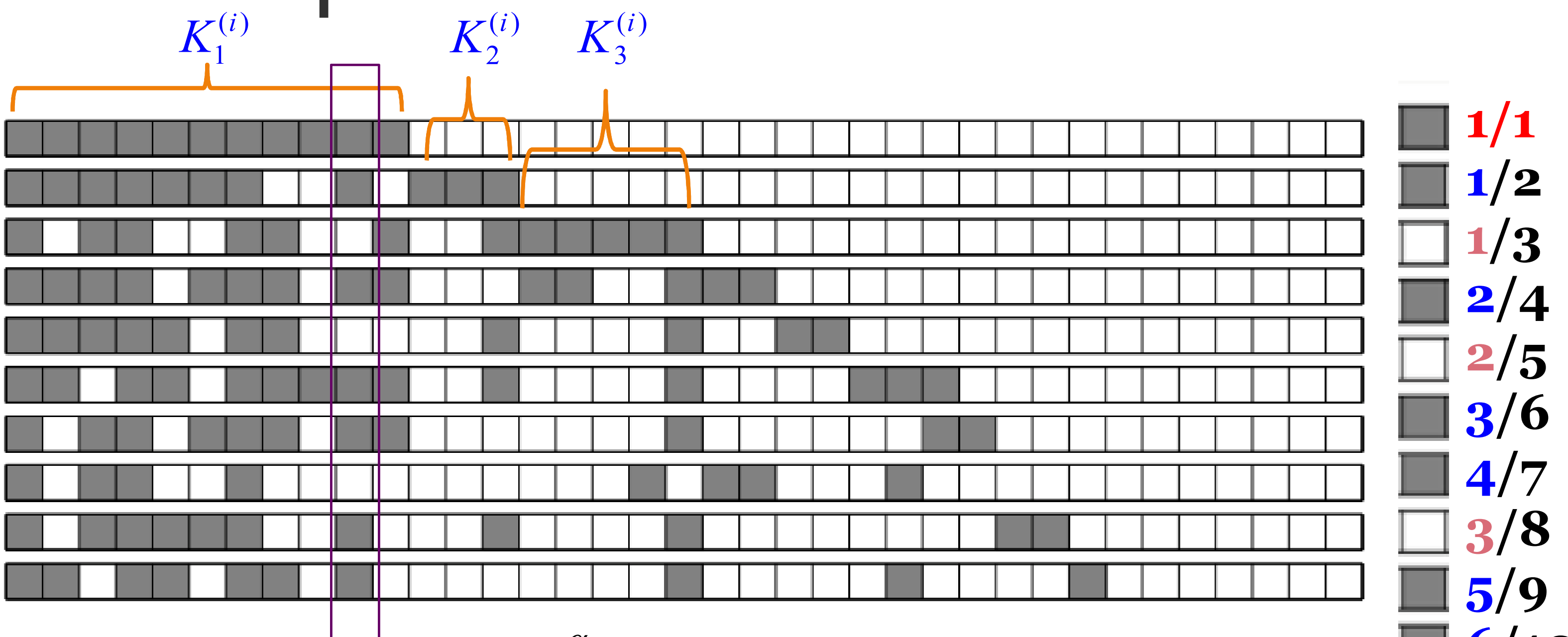

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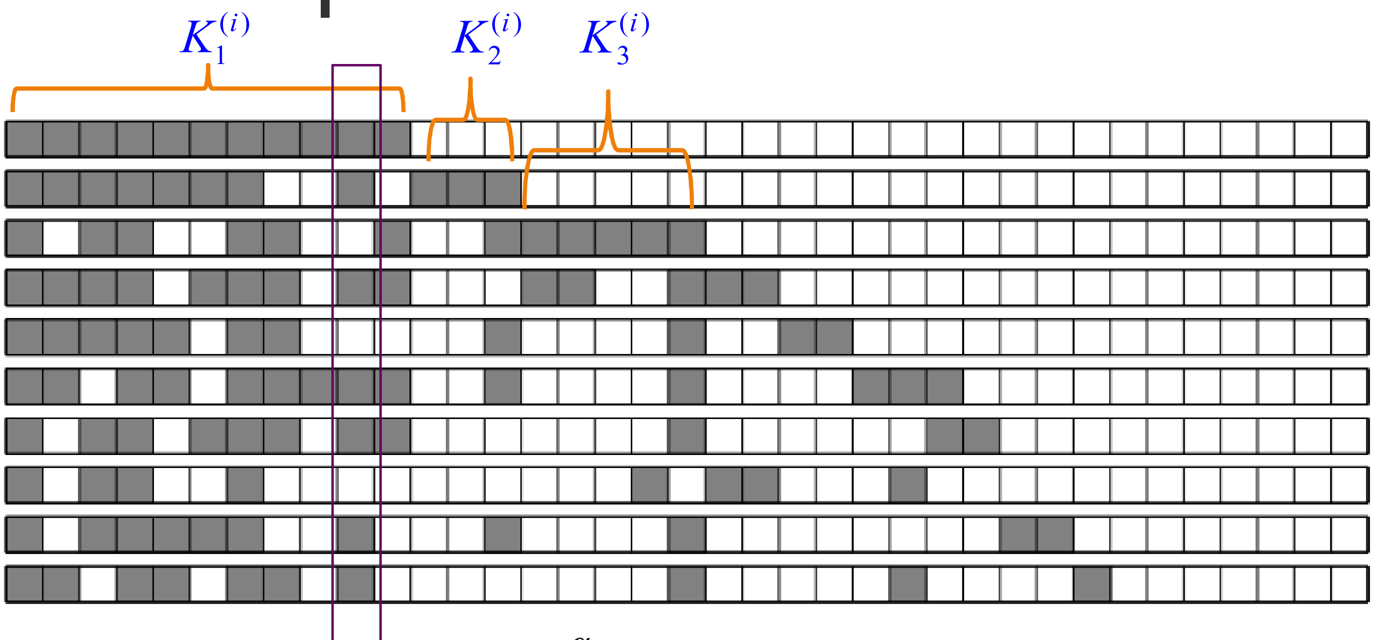

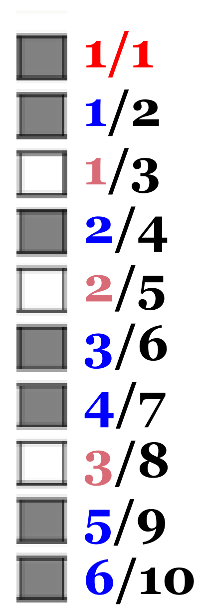

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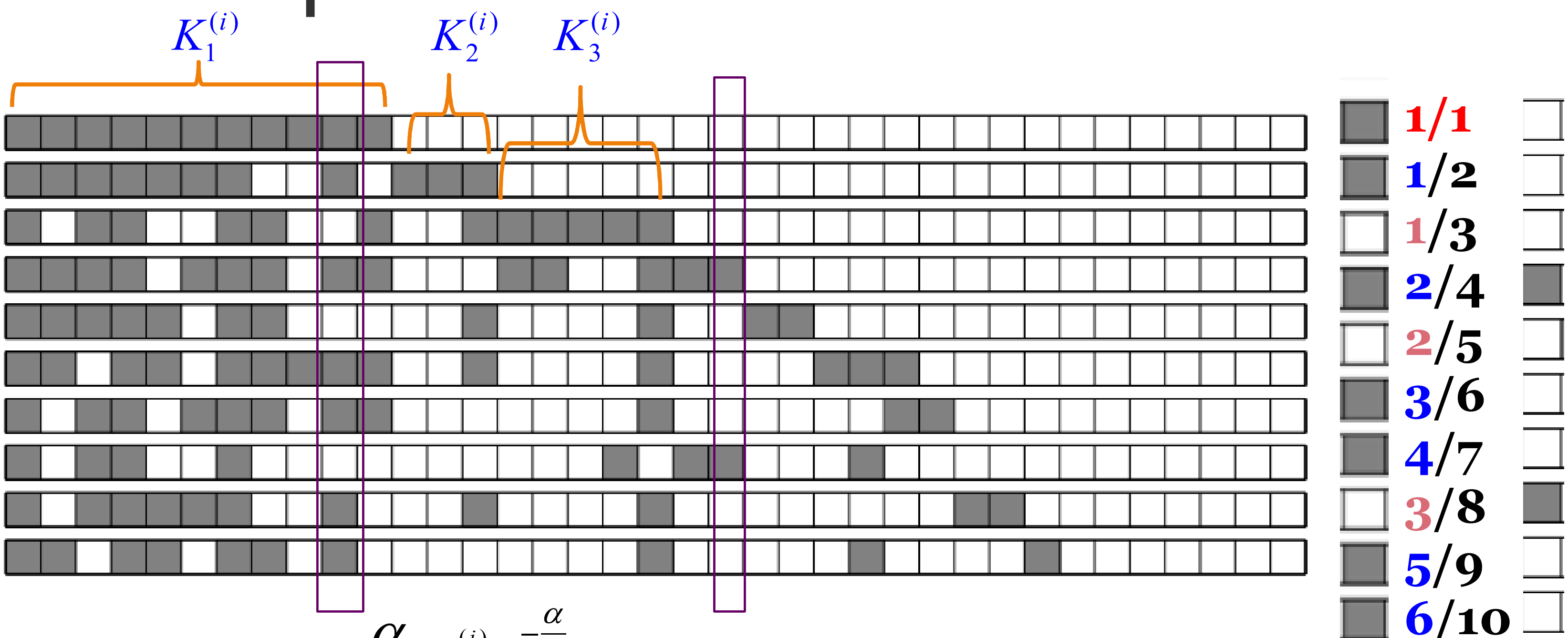
$$\frac{3!6!}{10!}$$

Indian Buffet Process

2st Representation: Customers & Dishes



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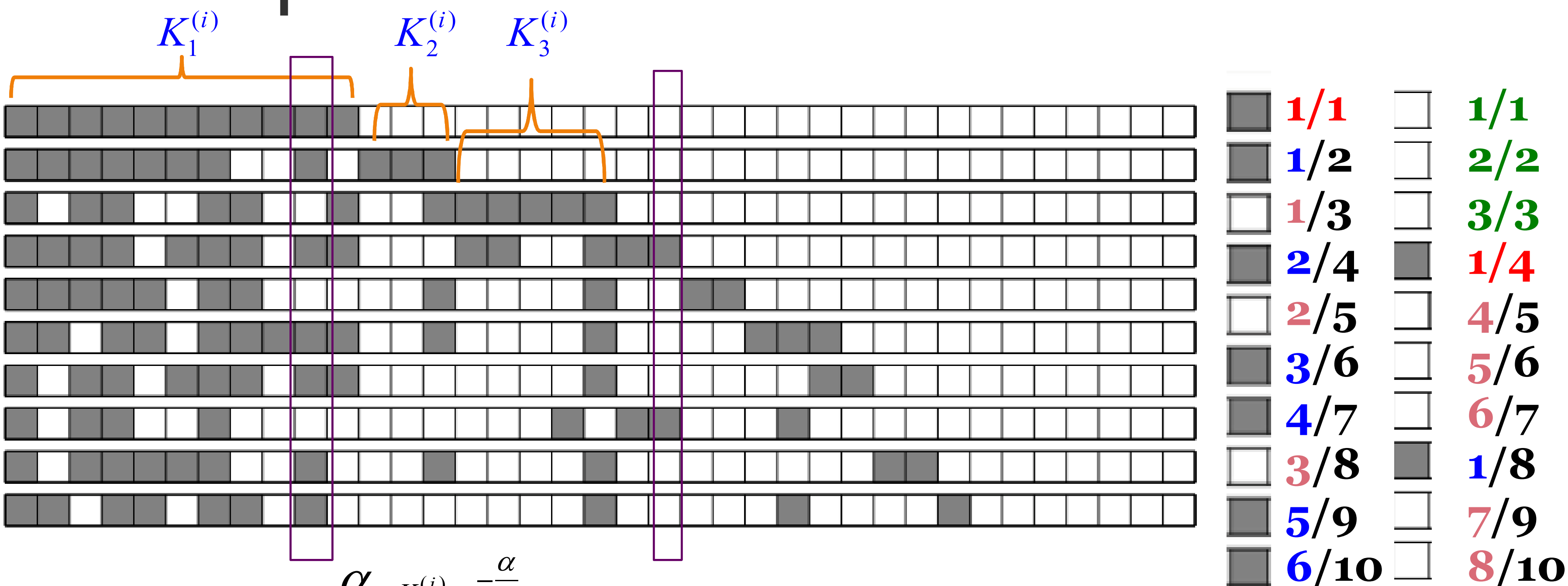

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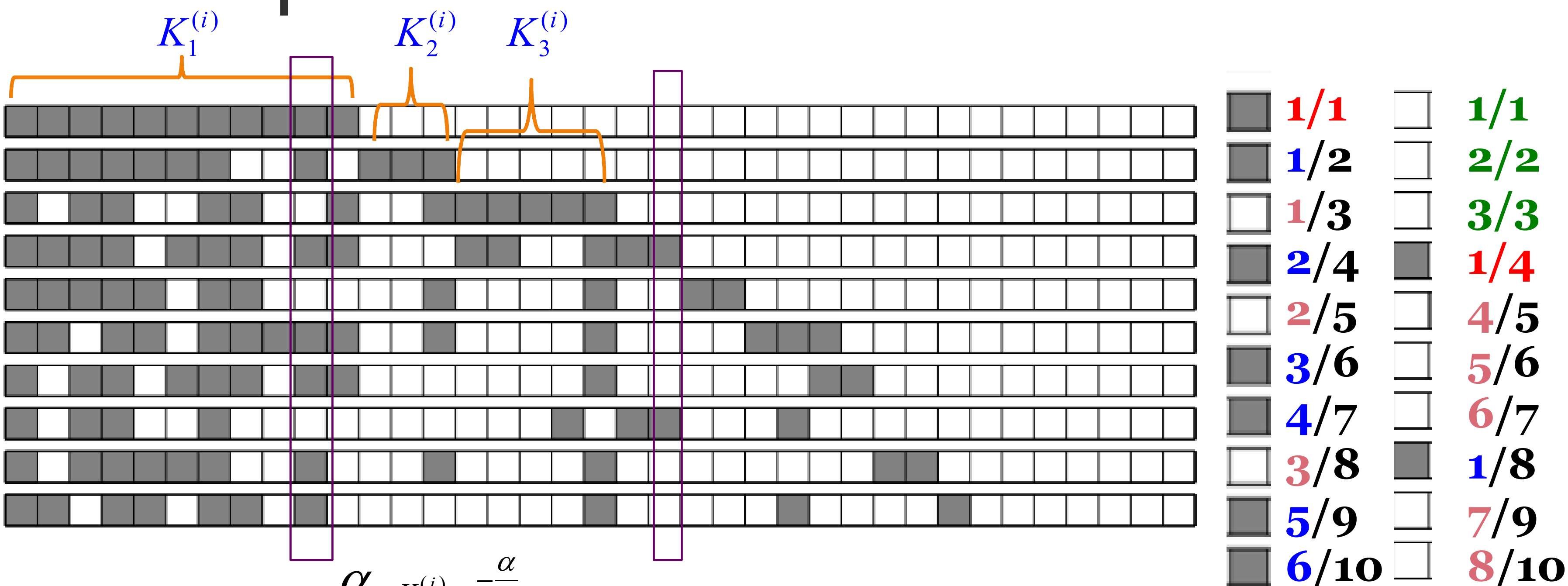
Indian Buffet Process

2st Representation: Customers & Dishes



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Indian Buffet Process
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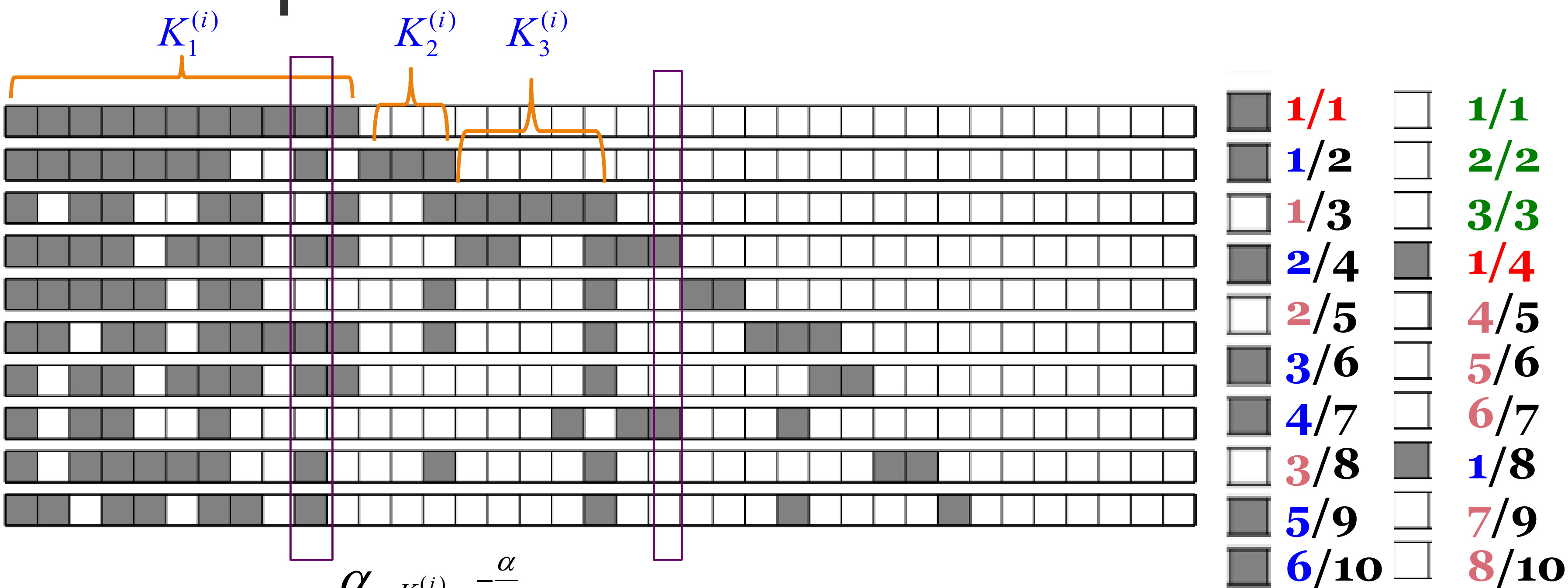

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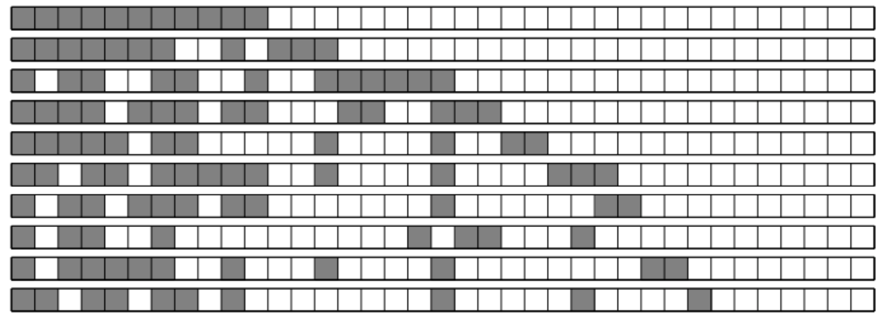
Indian Buffet Process
2st Representation: Customers & Dishes



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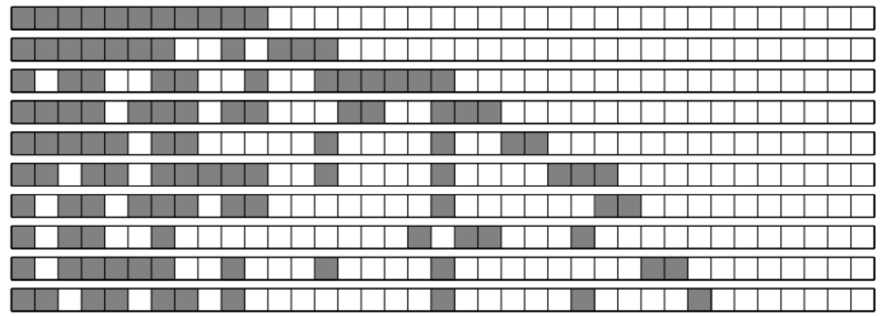
$$P_{IBP}(Z \mid \alpha) = \frac{\alpha^{K_+} e^{-\alpha H_N}}{\prod_{i}^{N} K_1^{(i)}} \prod_{k}^{K_+} \frac{(N-m_k)!(m_k-1)!}{N!}$$

Indian Buffet Process
2st Representation: Customers & Dishes



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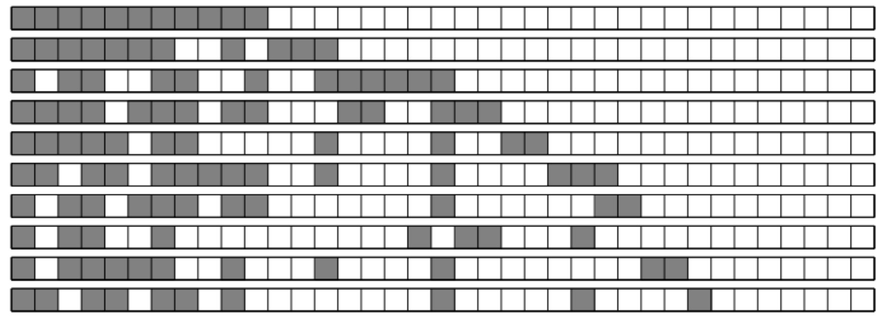

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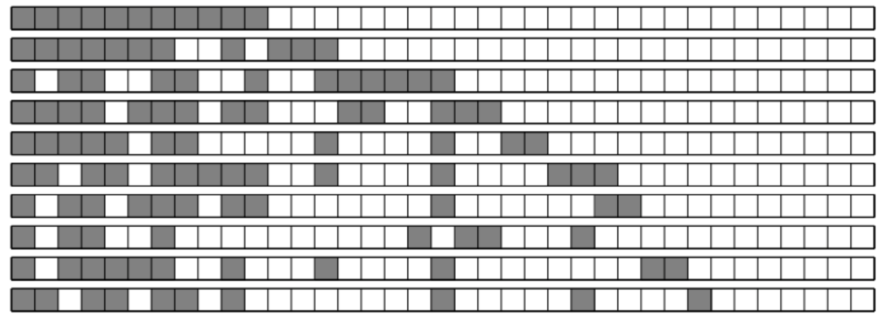

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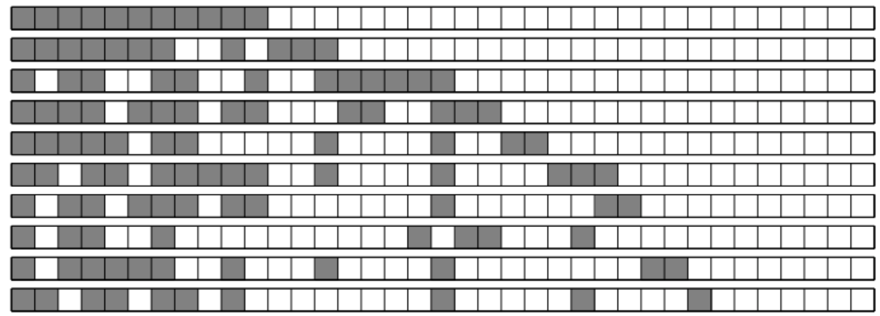

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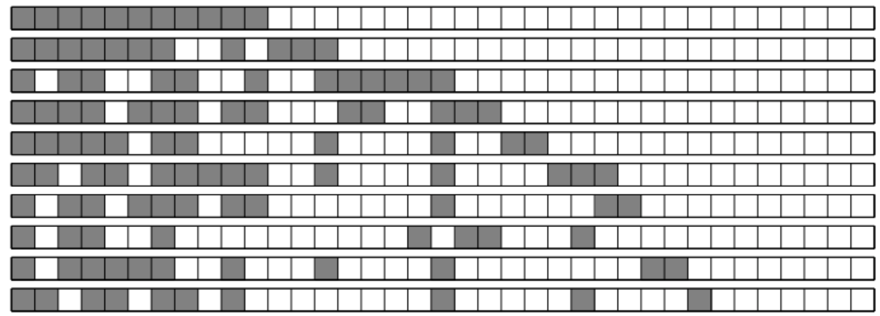

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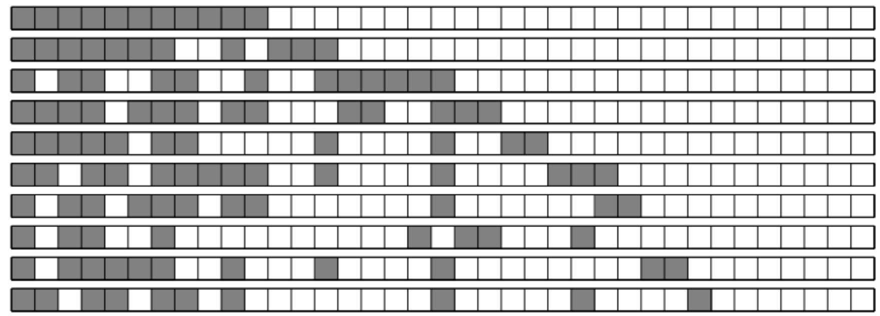

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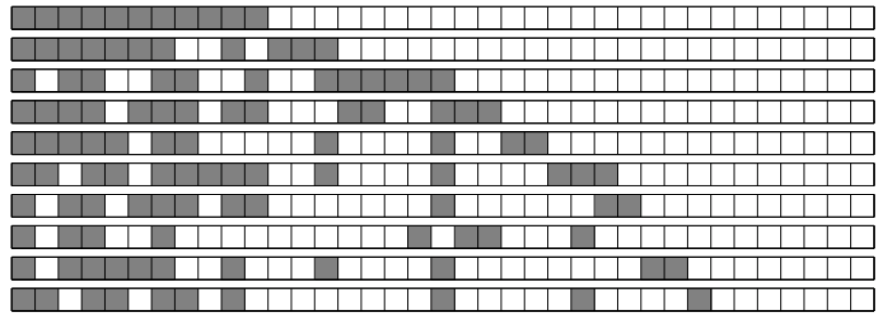

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- Number of permutation: $\dfrac{\prod_{i}^{N} K_1^{(i)}}{\prod_{h=1}^{2^N-1} K_h!}$ ⇒ $P_{IBP}([Z] \mid \alpha) = \dfrac{\alpha^{K_+} e^{-\alpha H_N}}{\prod_{h=1}^{2^N-1} K_h!} \prod_{k}^{K_+} \dfrac{(N-m_k)!(m_k-1)!}{N!}$

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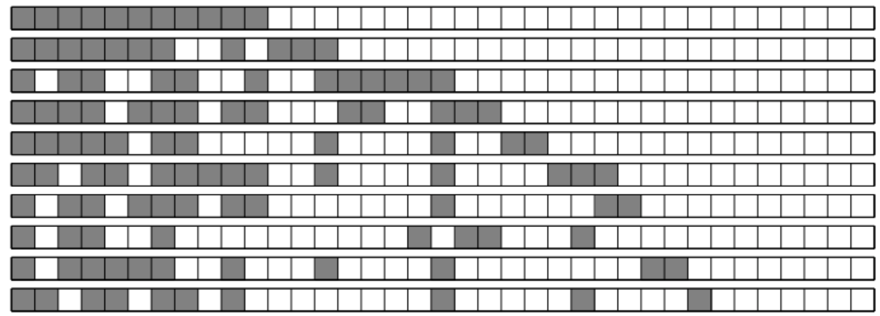

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Note:

- Permute $K_1^{(1)}$, next $K_1^{(2)}$, next $K_1^{(3)}$,… next $K_1^{(N)}$ dishes (columns) does not change P(Z)
- The permuted matrices are all of same *lof* equivalent class [Z]
- Number of permutation: $\frac{\prod_i^N K_1^{(i)}}{\prod_{h=1}^{2^N-1} K_h!}$

$$\Rightarrow P_{IBP}([Z] \mid \alpha) = \frac{\alpha^{K_+} e^{-\alpha H_N}}{\prod_{h=1}^{2^N-1} K_h!} \prod_k^{K_+} \frac{(N-m_k)!(m_k-1)!}{N!}$$

2^{nd} representation is equivalent to 1^{st} representation

Indian Buffet Process

3rd Representation: **Distribution over Collections of Histories**

- Directly generating the left ordered form (*lof*) matrix **Z**
- For each history h:
  - m_h: number of non-zero elements in h
  - Generate K_h columns of history h

$$K_h \sim \text{Poission}(\alpha \frac{(m_h - 1)!(N - m_h)!}{N!})$$

- The distribution over collections of histories

$$P(\mathbf{K}) = \prod_{h=1}^{2^N-1} \frac{\left(\alpha \frac{(m_h-1)!(N-m_h)!}{N!}\right)^{K_h}}{K_h!} \exp\left\{-\alpha \frac{(m_h-1)!(N-m_h)!}{N!}\right\}$$

$$= \frac{\alpha^{\sum_{h=1}^{2^N-1} K_h}}{\prod_{h=1}^{2^N-1} K_h!} \exp\{-\alpha H_N\} \prod_{h=1}^{2^N-1} \left(\frac{(m_h-1)!(N-m_h)!}{N!}\right)^{K_h},$$

- **Note:**
  - **Permute digits in h does not change m_h (nor P(K))**
  - **Permute rows means customers are exchangeable**

Indian Buffet Process

- Effective dimension of the model K_+:
  - Follow Poission distribution: $K_+ \sim \text{Poission}(\alpha H_N)$
  - Derives from 2nd representation by summing Poission components
- Number of features possessed by each object:
  - Follow Possion distribution: $\text{Poission}(\alpha)$
  - Derives from 2nd representation:
    - The first customer chooses $\text{Poission}(\alpha)$ dishes
    - The customers are exchangeable and thus can be purmuted
- $\mathbf{Z}$ is sparse:
  - Non-zero element
  - Derives from the 2nd (or 1st) representation: α
    - Expected number of non-zeros for each row is $N\alpha$
    - Expected entries in $\mathbf{Z}$ is

IBP: Gibbs sampling

- Need to have the full conditional

$$p(z_{ik}=1\mid \mathbf{Z}_{-(ik)},\mathbf{X}) \propto p(\mathbf{X}\mid \mathbf{Z})\,p(z_{ik}=1\mid \mathbf{Z}_{-(ik)})$$

  - $\mathbf{Z}_{-(n,k)}$ denotes the entries of ***Z*** other than z_{nk} .
  - p(***X***|***Z***) depends on the model chosen for the observed data.
- By exchangeability, consider generating row as the last customer:
- By IBP, in which $p(z_{ik}=1\mid \mathbf{z}_{-ik}) = \dfrac{m_{-i,k}}{N}$
  - If sample z_{ik}=0, and m_k=0: delete the row
  - At the end, draw new dishes from $\text{Pois}(\frac{\alpha}{n})$ with considering $p(\mathbf{X}\mid \mathbf{Z})$
    - Approximated by truncation, computing probabilities for a range of values of new dishes up to a upper bound

More talk on applications

- Applications
  - As prior distribution in models with infinite number of features.
  - Modeling Protein Interactions
  - Models of bipartite graph consisting of one side with undefined number of elements.
  - Binary Matrix Factorization for Modeling Dyadic Data
  - Extracting Features from Similarity Judgments
  - Latent Features in Link Prediction
  - Independent Components Analysis and Sparse Factor Analysis
- More on inference
  - Stick-breaking representation (Yee Whye Teh et al., 2007)
  - Variational inference (Finale Doshi-Velez et al., 2009)
  - Accelerated Inference (Finale Doshi-Velez et al., 2009)

References

- Griffiths, T. L., & Ghahramani, Z. (2011). The indian buffet process: An introduction and review. *Journal of Machine Learning Research*, *12*, 1185-1224.
- Slides: Tom Griffiths, *"The Indian buffet process"*.