CS 598CSC: Combinatorial Optimization	**Lecture date: May 4th, 2010**
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1 Okamura-Seymour Theorem

Theorem 1 *Let $G = (V, E)$ be a plane graph and let $H = (T, R)$ be a demand graph where T is the set of vertices of a single face of G. Then if G satisfies the cut condition for H and $G + H$ is eulerian, there is an integral multiflow for H in G.*

The proof is via induction on $2|E| - |R|$. Note that if G satisfies the cut condition for H, then $|R| \leq |E|$ (why?).

There are several "standard" induction steps and observations that are used in this and other proofs and we go over them one by one. For this purpose we assume G, H satisfy the conditions of the theorem and is a counter example with $2|E(G)| - |R|$ minimal[1].

Claim 2 *No demand edge r is parallel to a supply edge e.*

Proof: If r is parallel to e then $G - e$, $H - r$ satisfy the conditions of the theorem and by induction $H - r$ has an integral multiflow in $G - e$. We can route r via e. Thus H has an integral multiflow in G. □

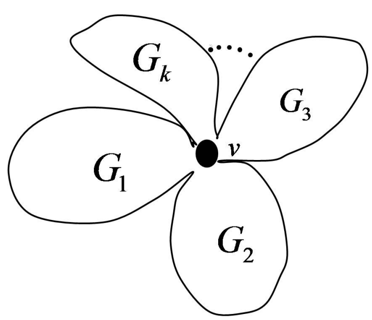
Definition 3 *A set $S \subset V$ is said to be tight if $|\delta_G(S)| = |\delta_H(S)|$.*

Claim 4 *For every demand edge $r \in R$ there is a tight cut S s.t. $r \in \delta_H(S)$.*

Proof: If r is not in any tight set, then adding two copies of r to H maintains cut condition and the Eulerian condition. By induction (note that the induction is on $2|E| - |R|$) the new instance is routable. □

Claim 5 *G is 2-node connected.*

Proof: Suppose not and let v be a cut vertex of G. Let $G_1, G_2, ..., G_k$ be the graphs obtained by combining v with the components of $G - v$.



Suppose there is a demand edge $r = (s, t)$ s.t. $s \neq v, t \neq v$ and $s \in G_i$ and $t \in G_j, i \neq j$. Then we can replace (s, t) by two edges (s, v) and (v, t). The claim is that this new instance satisfies the cut condition - we leave the formal proof as an exercise. Clearly Euler condition is maintained. The new instance is routable by induction which implies that the original instance is also routable.

If no such demand edge exits then all demand edges have both end points in G_i for some i. Then let H_i be the demand graph induced on G_i. We can verify that each G_i, H_i satisfy the cut condition and the Euler condition. By induction each H_i is routable in G_i which implies that H is routable in G. $\square$

Definition 6 *A set $\emptyset \subset S \subset V$ is central if $G[S]$ and $G[V \setminus S]$ are connected.*

Lemma 7 *Let G be a connected graph. Then G, H satisfy the cut condition if and only if the cut condition is satisfied for each central set S.*

Proof: Clearly, if G, H satisfy the cut condition for all sets then it is satisfied for the central sets. Suppose the cut condition is satisfied for all central sets but there is some non-central set S' such that $|\delta_G(S)| < |\delta_H(S)|$. Choose S' to be minimal among all such sets. We obtain a contradiction as follows. Let $S_1, S_2, \ldots, S_k$ be the connected components in $G \setminus \delta_G(S')$; since S' is not central, $k \geq 3$. Moreover each S_i is completely contained in S' or in $V \setminus S'$. We claim that some S_j violates the cut-condition, whose proof we leave as an exercise. Moreover, by minimality in the choice of S', S_j is central, contradicting the assumption. $\square$

One can prove the following corollary by a similar argument.

Corollary 8 *Let G, H satisfy the cut condition. If S' is a tight set and S' is not central, then there is some connected component S contained in S' or in $V \setminus S'$ such that S is a tight central set.*

Uncrossing:

Lemma 9 *Let G, H satisfy cut-condition, Let A, B be two tight sets such that $A \cap B \neq \emptyset$ and $A \cup B \neq V$. If $|\delta_H(A)| + |\delta_H(B)| \leq |\delta_H(A \cap B)| + |\delta_H(A \cup B)|$, then $A \cap B$ and $A \cup B$ are tight. If $|\delta_H(A)| + |\delta_H(B)| \leq |\delta_H(A - B)| + |\delta_H(B - A)|$, then $A - B$ and $A - B$ are tight.*

Proof: By submodularity and symmetry of the cut function $|\delta_G| : 2^V \to \mathbb{R}_+$, we have

$$|\delta_G(A)| + |\delta_G(B)| \geq |\delta_G(A \cap B)| + |\delta_G(A \cup B)|$$

and also

$$\delta_G(A)| + |\delta_G(B)| \geq |\delta_G(A - B)| + |\delta_G(B - A)|.$$

Now if

$$|\delta_H(A)| + |\delta_H(B)| \leq |\delta_H(A \cap B)| + |\delta_H(A \cup B)|$$

then we have

$$|\delta_G(A \cap B)| + |\delta_G(A \cup B)| \geq |\delta_H(A \cap B)| + |\delta_H(A \cup B)| \geq |\delta_H(A)| + |\delta_H(B)| = |\delta_G(A)| + |\delta_G(B)|$$

where the first inequality follows from the cut-condition, the second from our assumption and the third from the tightness of A and B. It follows that

$$|\delta_G(A \cap B)| = |\delta_H(A \cap B)|$$

and

$$|\delta_G(A \cup B)| = |\delta_H(A \cup B)|.$$

The other claim is similar. $\square$

Corollary 10 *If A, B are tight sets and $\delta_H(A-B, B-A) = \emptyset$ then $A \cap B$ and $A \cup B$ are tight.*

Proof: We note that

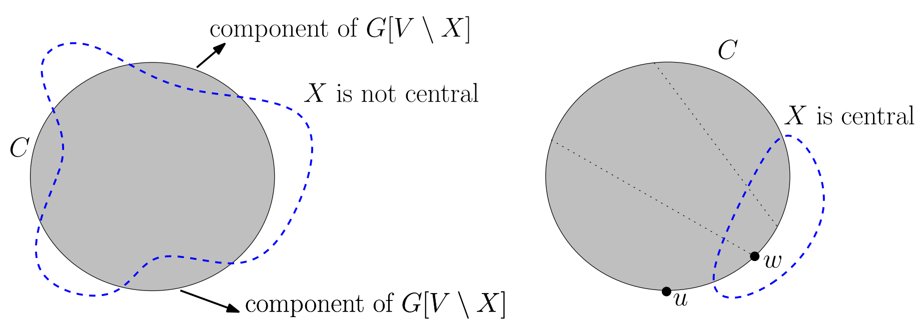
$$|\delta_H(A)| + |\delta_H(B)| = |\delta_H(A \cap B)| + |\delta_H(A \cup B)| + 2|\delta_H(A-B, B-A)|.$$

Thus, if $\delta_H(A-B, B-A) = \emptyset$ we have $|\delta_H(A)| + |\delta_H(B)| = |\delta_H(A \cap B)| + |\delta_H(A \cup B)|$ and we apply the previous lemma. □

Proof: Now we come to the proof of the Okamura-Seymour theorem. Recall that G, H is a counter example with $2|E| - |R|$ minimal. Then we have established that:

1. G is 2-connected.
2. every demand edge is in a tight cut.
3. no supply edge is parallel to a demand edge.

Without loss of generality we assume that the all the demands are incident to the outer/unbounded face of G. Since G is 2-connected the outer face is a cycle C. Let $X \subset V$ be a tight set; a tight set exists since each demand edge is in some tight set. Then if $X \cap C$ is not a contiguous segment, X is not a central set as can be seen informally by the picture below; $G[V \setminus X]$ would have two or more connected components.



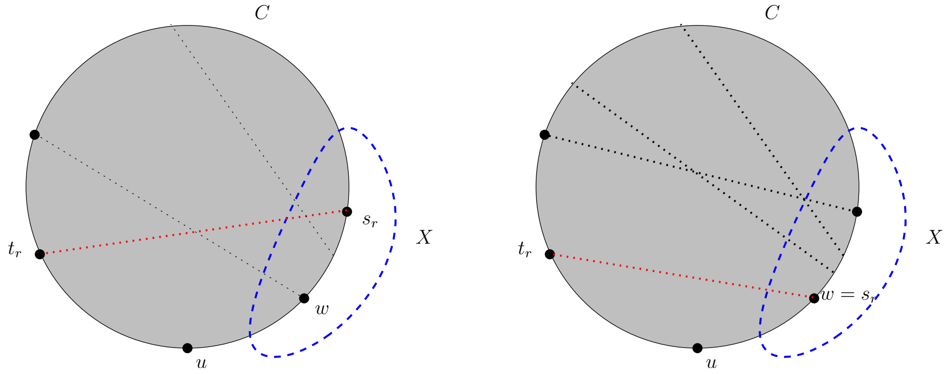
From Corollary 8 we can assume the existence of a tight set X such that $X \cap C$ is a contiguous segment. Choose such a tight set with $X \cap C$ minimal.

Let uw be one of the two edges of the cycle C that crosses X; let $w \in X$ and $u \notin X$. Since X is tight, $\delta_R(X) \neq \emptyset$. For each $r \in \delta_R(X)$, let s_r, t_r be the endpoints of r with $s_r \in X \cap C$ and $t_r \notin X \cap C$. Choose $r \in \delta_R(X)$ such that t_r is closest (in distance along the cycle C) to u in $C - X$. Note that r is not parallel to uw. So if $s_r = w$ then $t_r \neq u$ and if $t_r = u$ then $s_r \neq w$. Let $v \in \{u, w\} \setminus \{s_r, t_r\}$, v exists by above; for simplicity choose $v = w$ if $s_r \neq w$.

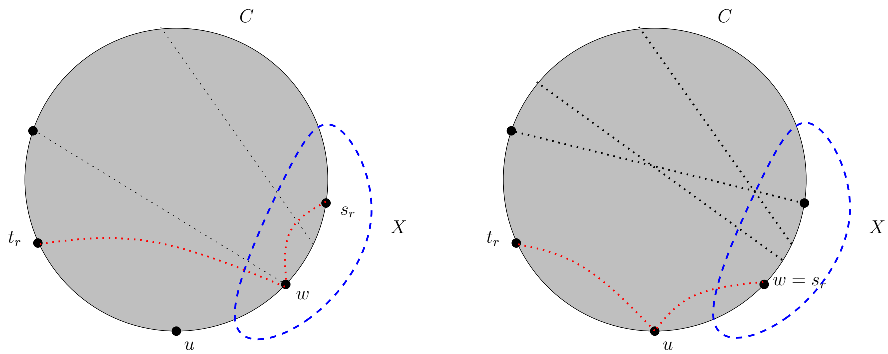
Let $R' = (R \setminus \{s_r, t_r\}) \cup \{s_r v, v t_r\}$. That is, we replace the demand edge $s_r t_r$ by two new demand edges $s_r v$ and $v t_r$ as shown in the figure.

Claim 11 *G satisfies cut condition for R' and $E + R'$ induces an Eulerian graph.*

Assuming claim, we are done because $2|E| - |R'| < 2|E| - |R|$ and by induction R' has an integral multiflow in G, and R has an integer multiflow if R' does.



In the picture on the left $v = w$ and on the right $v = u$.



Replacing $s_r t_r$ by new demands $s_r v$ and $v t_r$.

Trivial to see $E + R'$ induces an Eulerian graph. Suppose G does not satisfy the cut condition for the demand set R'. Let Y be a cut that violates the cut condition for R'. For this to happen Y must be a tight set for R in G; this is the reason why replacing $s_r t_r$ by $s_r v$ and $v t_r$ violates the cut condition for Y for R'. By complementing Y if necessary we can assume that $v \in Y, s_r, t_r \notin Y$. Further, by Corollary 8, we can assume Y is central and hence $Y \cap C$ is a contiguous segment of C.

By choice of r there is no demand r' between $Y - X$ and $X - Y$. If there was, then $t_{r'}$ would be closer to u than t_r. We have X, Y tight and

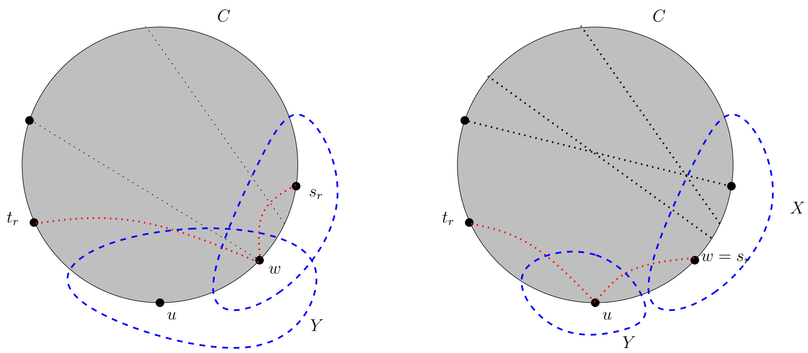
$$\delta_R[X - Y, Y - X] = \emptyset.$$

We consider two cases. First, suppose $X \cap Y \neq \emptyset$ (this is guaranteed if $v = w$). Then from Corollary 10, $X \cap Y$ and $X \cup Y$ are tight since $X \cap Y \neq \emptyset$ by assumption and $X \cup Y \neq V$ (since $t_r \in V \setminus (X \cup Y)$). $X - Y \neq \emptyset$, since $s_r \in X - Y$. Since $X \cap Y$ is a tight set and $X \cap Y \neq X$, it contradicts the choice of X as the tight set with $X \cap C$ minimal. If $X \cap Y = \emptyset$ then $v = u$ and $u \in Y$; again $X \cup Y \neq V$. Note that the edge uw joins X and Y. In this case we claim that $X \cup Y$ does not satisfy the cut condition which is a contradiction. To see this note that

$$|\delta_G(X \cup Y)| \leq |\delta_G(X)| + |\delta_G(Y)| - 2$$

since uw connects X to Y. However,

$$|\delta_H(X \cup Y)| = |\delta_H(X)| + |\delta_H(Y)| = |\delta_G(X)| + |\delta_G(Y)|$$



Tight set Y in the two cases.

where the first inequality follows since $X \cap Y = \emptyset$ and there are no demand edges between $X - Y$ and $Y - X$. The second inequality follows from the tightness of X and Y. $\square$

2 Sparse Cuts, Concurrent Multicomodity Flow and Flow-Cut Gaps

In traditional combinatorial optimization, the focus has been on understanding and characterizing those cases where cut condition implies existence of fractional/integral multiflow. However, as we saw, even in very restrictive settings, cut condition is not sufficient. A theoretical CS/algorithms perspective has been to quantify the "gap" between flow and cut. More precisely, suppose G satisfies the cut condition for H. Is it true that there is a feasible multiflow in G that routes λd_i for each pair $s_i t_i$ where λ is some constant in $(0, 1)$?

There are two reasons for considering the above. First, it is a mathematically interesting question. Second, and this was the initial motivation from a computer science/algorithmic point of view, is to obtain approximation algorithms for finding "sparse" cuts in graphs; these have many applications in science and engineering. The following is known.

Theorem 12 *Given a multiflow instance, it is co-NP complete to check if the cut-condition is satisfied for the instance.*

Definition 13 *Given a multiflow instance the maximum concurrent flow for the given instance is the maximum $\lambda \geq 0$ such that there is a feasible multiflow if all demand values are multiplied by λ.*

Proposition 14 *There is a polynomial time algorithm that, given a multiflow instance, computes the maximum concurrent flow.*

Proof: Write a linear program:

$$\begin{aligned} &\max \lambda \\ \text{flow for each} \quad &s_i t_i \geq \lambda d_i \end{aligned}$$

Flow satisfies capacity constraints. We leave the details to the reader. $\square$

Definition 15 *Given a multiflow instance on G, H, the sparsity of a cut $U \subset V$ is*

$$sparsity(U) := \frac{c\,(\delta_G(U))}{d\,(\delta_H(U))}.$$

A sparsest cut is $U \subset V$ such that $sparsity(U) \leq sparsity(U')$ for all $U' \subset V$. We refer to $\min_{U \subset V} sparsity(U)$ as the min-sparsity of the given multiflow instance.

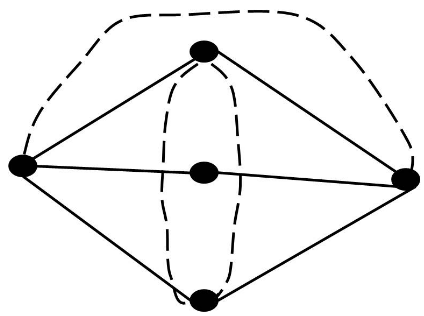
Observation 16 *(G, H) satisfies the cut condition implies sparsity$(U) \geq 1$ for all $\mathcal{U} \subset V$.*

Proposition 17 *In many multiflow instance, if λ^* is the max concurrent flow then*

$$\lambda^* \leq sparsity(U), \forall U \subset V.$$

The ratio $\frac{min-sparity}{\lambda^*}$ is the flow cut gap for the given instance.

For example,



with capacities and demands equal to 1, the flow-cut gap is $\frac{4}{3}$. Min-sparsity for the above instance is 1 while $\lambda^* = \frac{3}{4}$. In general, we are interested in quantifying flow-cut gaps for classes of instances rather than a particular instance.

In the sequel, we think of G and H as "topological" graphs in that they are not endowed with capacities and demands. A multiflow instance on G, H is defined by $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. Note that by setting $c(e) = 0$ or $d(r) = 0$, we can "eliminate" some edges. We define $\alpha(G, H)$, the flow-cut gap for G, H, as the supremum over all instances on G, H defined by capacities $c : E \to \mathbb{R}_+$ and $d : R \to \mathbb{R}_+$. We can then define for a graph G:

$$\alpha(G) = \sup_{\substack{H=(T,R) \\ T \subseteq V}} (G, H).$$

Some results that we mentioned on the sufficiency of cut condition for feasible flow can be restated as follows: $\alpha(G, H) = 1$ if $|R| = 2$ (Hu's theorem), $\alpha(G, H) = 1$ if G is planar and T is the vertex set of a face of G (Okamura-Seymour theorem), and so on. What can we say about $\alpha(G)$ for an arbitrary graph?

Theorem 18 (Linial-London-Rabinovich, Aumann-Rabani) *$\alpha(G) = O(\log n)$ where $n = |V|$ and in particular $\alpha(G, H) = O(\log |R|)$ i.e. the flow-cut gap is $O(\log k)$ for k-commodity flow. Moreover there exist graphs G, H for which $\alpha(G, H) = \Omega(\log |R|)$, in particular there exist graphs G for which $\alpha(G) = \Omega(\log n)$.*

Conjecture 19 *$\alpha(G) = O(1)$ if G is a planar graph.*

Theorem 20 (Rao) *$\alpha(G) = O(\sqrt{\log n})$ for a planar graph G.*

References

[1] Lex Schrijver, "Combinatorial Optimization: Polyhedra and Efficiency", Chapter 74, Vol. C, Springer-Verlag, 2003.