CS 598CSC: Combinatorial Optimization **Lecture date: April 29, 2010**
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1 Multiflows

The maxflow-mincut theorem of Ford and Fulkerson generalizes Menger's theorem and is a fundamental result in combinatorial optimization with many applications.

Theorem 1 *In a digraph $D = (V, A)$ with arc capacity function $c : A \to \mathbb{R}_+$, the maximum s-t flow value is equal to the minimum s-t capacity cut value. Moreover, if c is integer valued, then there is an integer valued maximum flow.*

In particular, the maximum number of s-t arc-disjoint paths in a digraph is equal to the minimum number of arcs whose removal disconnects s from t (Menger's theorem). When applied to undirected graphs we obtain the edge-disjoint and node-disjoint path version of Menger's Thoerem.

In many applications in networks we are interested in multiflows, also referred to as multicommodity flows. $s - t$ flows are also referred to as single-commodity flows.

A *multiflow instance* in a directed graph consists of a directed "supply" graph $D = (V, A)$ with non-negative arc capacities $c : A \to \mathbb{R}_+$ and a demand graph $H = (T, R)$ with $T \subseteq V$ called terminals, and non-negative demand requirements $d : R \to \mathbb{R}_+$. The arcs in R are referred to as nets. The demand graph can also be specified as a set of ordered pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with $d_i \in \mathbb{R}_+$ denoting the demand for (s_i, t_i). This is referred to as the k-commodity flow problem.

A *multiflow instance* in an undirected graph consists of an undirected supply graph $G = (V, E)$ and an undirected demand graph $H = (T, R)$. The demand graph can be specified by a collection of unordered pairs $s_1t_1, \ldots, s_kt_k$.

Given a multiflow instance in a directed graph $D = (V, A)$ with demand graph $H = (T, R)$, a *multiflow* is a collection of flows, $f_r, r \in R$ where f_r is an s_r-t_r flow and $r = (s_r, t_r)$. A multiflow satisfies the capacity constraints of the supply graph if for each arc $a \in A$,

$$\sum_{r \in R} f_r(a) \leq c(a). \tag{1}$$

The multiflow satisfies the demands if for each $r = (s_r, t_r) \in R$, the f_r flow from s_r to t_r is at least $d(r)$.

For undirected graphs we need a bit more care. We say that $f : E \to \mathbb{R}_+$ is an $s - t$ flow if there is an orientation $D = (V, A)$ of $G = (V, E)$ such that $f' : A \to \mathbb{R}_+$ defined by the orientation and $f : E \to \mathbb{R}_+$ is an $s - t$ flow in D. Thus $f_r,\ r \in R$ where $f_r : E \to \mathbb{R}_+$ is a multiflow if each f_r is an s_r-t_r flow. It satisfies the capacity constraints if $\forall e \in E$,

$$\sum_{r \in R} f_r(e) \leq c(e). \tag{2}$$

We say a multiflow is *integral* if each of the flows is integer valued; that is $f_r(a)$ is an integer for each arc a and each $r \in R$. Similarly half-integral (i.e., each flow on an arc is an integer multiple of 1/2).

Proposition 2 *Given a multiflow instance in a directed graph, there is a polynomial time algorithm to check if there exists a multiflow that satisfies the capacities of the supply graph and the demand requirements of the demand graph.*

Proof: Can be solved by expressing the problem as a linear program. Variables $f_r(a)$ $r \in R,\ a \in A$. Write standard flow conservation constraints that ensures $f_r : A \to \mathbb{R}_+$ is a flow for each r (flow conservation at each node other than the source and destination of r). We add the following set of constraints to ensure capacity constraints of the supply graph are respected.

$$\sum_{r \in R} f_r(a) \leq c(a) \quad a \in A. \tag{3}$$

Finally, we add constraints that the value of f_r (leaving the source of r) should be at least $d(r)$. $\square$

Proposition 3 *Given an undirected multiflow instance, there is a polynomial time algorithm to check if there is a feasible multiflow that satisfies the supply graph capacities and the demand requirements.*

Proof: We reduce it to the directed flow case as follows. Given $G = (V, E)$ obtain a digraph $D = (V, A)$ by dividing each edge e into two arcs $\overrightarrow{e}$ and $\overleftarrow{e}$. Now we have variable $f_r(a),\ a \in A,\ r \in R$, and write constraints that ensure that $f_r : A \to \mathbb{R}_+$ is a flow of value $d(r)$ from s_r to t_r where $r = s_r t_r$. The capacity constraint ensures that the total flow on both $\overrightarrow{e}$ and $\overleftarrow{e}$ is at most $c(e)$, i.e.,

$$\sum_{r \in R} (f_r(\overrightarrow{e}) + f_r(\overleftarrow{e})) \leq c(e),\ e \in E. \tag{4}$$

$\square$

LP duality gives the following useful necessary and sufficient condition; it is some times referred to as the Japanese theorem.

Theorem 4 *A multiflow instance in directed graph is feasible iff*

$$\sum_{i=1}^{k} d_i \ell(s_i, t_i) \leq \sum_{a \in A} c(a)\ell(a) \tag{5}$$

for all length functions $\ell : A \to \mathbb{R}_+$.

Here $\ell(s_i, t_i)$ is the shortest path distance from s_i to t_i with arc lengths $\ell(a)$. For undirected graph the characterization is similar

$$\sum_{i=1}^{k} d_i \ell(s_i, t_i) \leq \sum_{e \in E} c(e)\ell(e) \tag{6}$$

for all $\ell : E \to \mathbb{R}_+$.

Proof: Consider the path formulation we prove it for undirected graphs. Let P_i be the set of $s_i t_i$ path in G. Let $f_i : P_i \to \mathbb{R}_+$ be an assignment of flow values to paths in P_i. We want feasibility of

$$\sum_{p \in P_i} f_i(p) \geq d_i \quad i = 1 \text{ to } k \tag{7}$$

$$\sum_{i=1}^{k} \sum_{p \in P_i : e \in p} f_i(p) \leq c(e) \quad e \in E \tag{8}$$

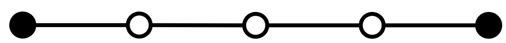
$$f_i(p) \geq 0, \quad p \in P_i, \quad 1 \leq i \leq k. \tag{9}$$

We apply Farkas lemma. Recal that $Ax \leq b,\ x \geq 0$ has a solution iff $yb \geq 0$ for each row vector $y \geq 0$ with $yA \geq 0$. We leave it as an exercise to derive the statement from Farkas lemma applied to the above system of inequalities. □

It is useful to interpret the necessity of the condition. Suppose for some $\ell : E \to \mathbb{R}_+$

$$\sum_{i=1}^{k} d_i l(s_i, t_i) > \sum_{e \in E} c(e)\ell(e) \tag{10}$$

we show that there is no feasible multiflow. For simplicity assume ℓ is integer valued. Then replace an edge e with length $\ell(e)$ by a path of length $\ell(e)$



and place capacity $c(e)$ on each edge. Suppose there is a feasible flow. For each (s_i, t_i), each flow path length is of length at least $l(s_i, t_i) \Rightarrow$ total capacity used up by flow for (s_i, t_i) is $\geq d_i\ell(s_i, t_i)$. But total capacity available is $\sum_e c(e)\ell(e)$ (after expansion). Hence if $\sum_{i=1}^{k} d_i\ell(s_i, t_i) > \sum_{e \in E} c(e)\ell(e)$, there cannot be a feasible multiflow.

To show that a multiflow instance is not feasible it is sufficient to give an appropriate arc length function that violates the necessary condition above.

2 Integer Multiflow and Disjoint Paths

When all capacities are 1 and all demands are 1 the problem of checking if there exists an integer multiflow is the same as asking if there exist arc-disjoint path (edge-disjoint path if graph is undirected) connecting the demand pairs.

The *edge-disjoint paths problem (EDP)* is the following decision problem: given supply graph $D = (V, A)$ (or $G = (V, E)$) and a demand graph $H = (T, R)$, are there arc/edge-disjoint paths connecting the pairs in R?

Theorem 5 (Fortune-Hopcroft-Wyllie 1980) *EDP in directed graphs is NP-complete even for two pairs.*

Theorem 6 *EDP in undirected graphs is NP-complete when $|R|$ is part of the input, even when $|R|$ consists of three sets of parallel edges.*

A deep, difficult and fundamental result of Robertson and Seymour is that EDP in undirected graphs is polynomial-time solvable when $|R|$ is fixed. In fact they prove that the vertex-disjoint path problem (the pairs need to be connected by vertex disjoint paths) is also tractable.

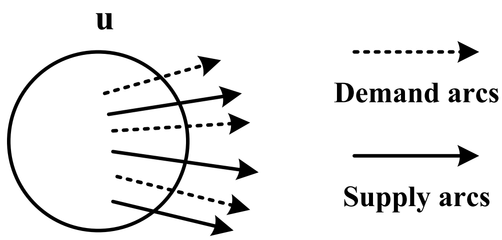
Theorem 7 (Robertson-Seymour) *The vertex-disjoint path problem is polynomial-time solvable if the number of demand pairs is a fixed constant.*

The above theorem relies on the work of Robertson and Seymour on graph minors.

3 Cut Condition—Sparsest Cuts and Flow-Cut Gaps

A necessary condition for the existence of a feasible multiflow for a given instance is the so called cut-condition. In directed graphs it is

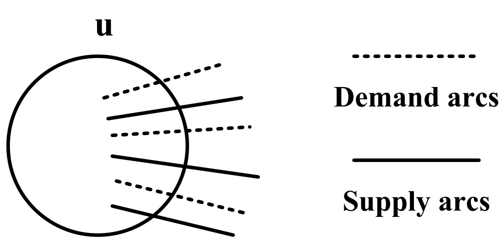
$$c(\delta_D^+(U)) \geq d(\delta_H^+(U)) \quad \forall U \subset V \tag{11}$$



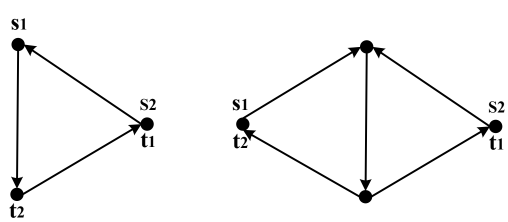
where $c(\delta_D^+(U))$ is capacity of all arcs leaving U, and $d(\delta_H^+(U))$ is the demand of all demand arcs leaving U. It is easy to see that this condition is necessary. Formally one sees that this condition is necessary by considering the length function $\ell : A \to \mathbb{R}_+$ where $\ell(a) = 1$ if $a \in \delta_D^+(U)$ and $\ell(a) = 0$.

For undirected graphs the cut condition states

$$c(\delta_G^+(U)) \geq d(\delta_H^+(U)) \quad \forall U \subset V \tag{12}$$

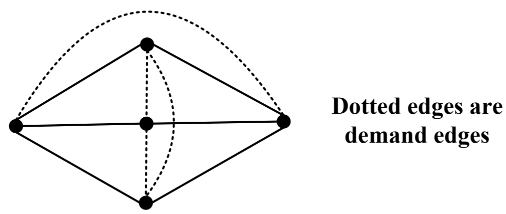


Cut condition is not sufficient in general. Consider the following examples in directed graphs



Cut condition is true for each case but no feasible multiflows exists as can be seen by considering the length function $\ell(a) = 1$ for each arc a.

For undirected graphs the following example is well known. Supply graph is $K_{2,3}$, a series-parallel graph. Again, cut-condition is satisfied but $\ell(e) = 1$ for each e shows no feasible multiflow exists.



4 Sufficiency of Cut Condition

Given that the cut condition is not sufficient for feasible flow it is natural to consider cases where it is indeed sufficient. First consider directed graphs. Suppose we have demand pairs of the form $(s, t_1), (s, t_2), \ldots, (s, t_k)$, i.e., all pairs share a common source. Then it is easy to see that cut condition implies feasible multiflow by reduction to the single-commodity flow case by connecting $t_1, t_2, \ldots, t_k$ to a common sink t. Similarly if the demand pairs are of the form $(s_1, t), (s_2, t), \ldots, (s_k, t)$ with a common source.

It turns out that these are the only interesting cases for which cut condition suffices. See Theorem 70.3 in Schrijver Book.

For undirected graphs several non-trivial and interesting cases where the cut-condition is sufficient are known. We list a few below:

- Hu's 2-commodity theorem shows that if there are only two pairs s_1t_1 and s_2t_2 then cut-condition is sufficient.
- Okamura-Seymour theorem states that if G is a planar graph and T is vertex set of a single face then cut condition is sufficient. Note that the theorem implies that for capacitated ring supply graphs the cut condition is sufficient.
- Okamura's theorem generalizes Okamura-Seymour Theorem. If G is planar and there are two faces F_1 and F_2 such that each $st \in R$ has both $s,\ t$ on one of the faces then cut condition is sufficient.
- Seymour's Theorem shows that if $G + H$ is planar then cut condition is sufficient.

For all of the above cases one has the following stronger result. If $G + H$ is Eulerian then the flow is guaranteed to be integral. To see that the Eulerian condition is necessary for integral flow in each of the above cases, consider the example below where the only feasible multiflow is a half-integral.