CS 598CSC: Combinatorial Optimization	**Lecture date: April 27th, 2010**
Instructor: Chandra Chekuri	**Scribe: Peixiang Zhao**

1 Submodular Flows

Network flows are a fundamental object and tool in combinatorial optimization. We have also seen submodular functions and their role in matroids, matroid intersection, polymatroids and polymatroid intersection. Edmonds and Giles developed the framework of submodular flows to find a common generalization of network flow and polymatroid intersection. Here is the model.

Definition 1 (Crossing Family) *Let $D = (V, A)$ be a directed graph and let $\mathscr{C} \subseteq 2^V$ be a family of subsets of V. $\mathscr{C}$ is called a **crossing family** if: $A, B \in \mathscr{C}$, $A \cap B \neq \emptyset$, $A \cup B \neq V \Rightarrow A \cap B \in \mathscr{C}$ and $A \cup B \in \mathscr{C}$.*

Definition 2 (Crossing Submodular) *Let $\mathscr{C}$ be a crossing family. A function $f : \mathscr{C} \to \mathbb{R}$ is called **crossing submodular** on $\mathscr{C}$ if it satisfies: $A, B \in \mathscr{C}$, $A \cap B \neq \emptyset$, $A \cup B \neq V \Rightarrow f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.*

Definition 3 (Submodular Flow) *Let $D = (V, A)$ be a digraph, $\mathscr{C}$ be a crossing family, and f be a crossing submodular function on $\mathscr{C}$. A vector $x \in \mathbb{R}^A$ is called a **submodular flow** if*

$$x(\delta^-(u)) - x(\delta^+(u)) \leq f(u), \qquad \forall u \in \mathscr{C} \tag{1}$$

Theorem 4 (Edmonds-Giles, 1977) *The system of inequalities shown in Equation 1 is box-TDI where $\mathscr{C}$ is a crossing family on V and f is crossing submodular on $\mathscr{C}$.*

Proof: We consider the primal-dual pair of LPs below where $w : A \to \mathbb{Z}_+$ and ℓ, u are integer vectors.

$$\begin{aligned} &\max \sum wx \\ \text{s.t.} \quad & x(\delta^-(U)) - x(\delta^+(U)) \leq f(U) \qquad U \in \mathscr{C} \\ & \ell \leq x \leq u \end{aligned}$$

and

$$\begin{aligned} &\min \sum_{U \in \mathscr{C}} f(U)y(U) + \sum_{a \in A} u(a)z_1(a) - \sum_{a \in A} \ell(a)z_2(a) \\ \text{s.t.} \quad & \sum_{U: U \in \mathscr{C}, a \in \delta^-(U)} y(U) - \sum_{U: U \in \mathscr{C}, a \in \delta^+(U)} y(U) + z_1(a) - z_2(a) = w(a) \qquad a \in A \\ & y, z_1, z_2 \geq 0 \end{aligned}$$

A family of sets $\mathcal{F} \subseteq 2^V$ is cross-free if for all $A, B \in \mathcal{F}$ the following holds:

$$A \subseteq B \text{ or } B \subseteq A \text{ or } A \cap B = \emptyset \text{ or } A \cup B = V.$$

Claim 5 *There exists an optimum solution y, z_1, z_2 such that $\mathscr{F} = \{U \in \mathscr{C} \mid y(U) > 0\}$ is cross-free.*

Proof: Suppose $\mathscr{F}$ is not cross-free. Then let $A, B \in \mathscr{F}$, such that $y(A) > 0$ and $y(B) > 0$ and $A \cap B \neq \emptyset$ and $A \cup B \neq V$. Then add $\epsilon > 0$ to $y(A \cup B)$, $y(A \cap B)$ and subtract $\epsilon > 0$ from $y(A)$ and $y(B)$. By submodularity of f, the objective function increases or remains same. We claim that alterting y in this fashion maintains dual feasibility; we leave this as an exercise.

By repeated uncrossing we can make $\mathscr{F}$ cross-free. Formally one needs to consider a potential function. For example, among all optimal solutions pick one that minimizes

$$\sum_{U \in \mathscr{C}} y(U)|U||V \backslash U|$$

□

Theorem 6 *Let $\mathscr{F}$ be a cross-free family on 2^V. Let M be an $|A| \times |\mathscr{F}|$ matrix where*

$$M_{a,U} = \begin{cases} 1 & \textit{if } a \in \delta^-(U) \\ -1 & \textit{if } a \in \delta^+(U) \\ 0 & \textit{otherwise} \end{cases}$$

Then M is TUM.

The proof of the above theorem proceeds by showing that M is a network matrix. See Schrijver Theorem 13.21 for details [1].

By the above one sees that the non-negative components of y, z_1, z_2 are determined by $[M, I, -I]$ and integer vector w where M is TUM. From this we infers that there exists an integer optimum solution to the dual. □

Corollary 7 *The polyhedron P determined by*

$$x(\delta^-(U)) - x(\delta^+(U)) \leq f(U) \qquad U \in \mathscr{C}$$

$$l \leq x \leq u$$

is an integer polyhedron whenever f is integer valued and l, u are integer vectors.

One can show that optimality on P can be done in strongly polynomial time if one has a value oracle for f. This can be done via a reduction to polymatroid intersection. We refer to Schrijver, Chapter 60 for more details [1].

2 Applications

Submodular flows are a very general framework as they combine graphs and submodular functions. We gave several applications below.

2.1 Circulations

Given a directed graph $D = (V, A)$, $x : A \to \mathbb{R}$ is a circulation if

$$x(\delta^-(v)) - x(\delta^+(v)) = 0, \qquad \forall v \in V$$

This can be modeled as a special case of submodular flow by setting $\mathscr{C} = \{\{v\} \mid v \in V\}$ and $f = 0$. We get the inequalities

$$x(\delta^-(v)) - x(\delta^+(v)) \le 0, \qquad v \in V.$$

One can check that the above inequalities imply that for any $\emptyset \subset U \subset V$ the inequality $x(\delta^-(U) - x(\delta^+(U) \le 0$ holds by adding up the inequalities for each $v \in U$. Combining this with the inequality $x(\delta^-(V \setminus U) - x(\delta^+(V \setminus U) \le 0$, we have $x(\delta^-(U) - x(\delta^+(U) = 0$ for all $\emptyset \subset U \subset V$ and in particular for each $v \in V$. The box-TDI result of submodular flow implies the basic results on circulations and flows including Hoffman's circulation theorem and the max-flow min-cut theorem.

2.2 Polymatroid Intersection

We saw earlier that the system

$$x(U) \le f_1(U) \qquad U \subseteq S$$
$$x(U) \le f_2(U) \qquad U \subseteq S$$

is box-TDI whenever f_1, f_2 are submodular functions on S. We can derive this from submodular flows as follows. Define S' and S'' are copies of S. Let $V = S' \uplus S''$ and define $\mathscr{C} = \{U' \mid U \subseteq S\} \cup \{S' \cup U'' \mid U \subseteq S\}$, where U' and U'' denote the sets of copies of elements of U in S' and S'' [1].

Claim 8 *$\mathscr{C}$ is a crossing family.*

Exercise 9 *Prove Claim 8.*

We further define $f : \mathscr{C} \to \mathbb{R}_+$ by

$$f(U') = f_1(U) \qquad U \subseteq S$$
$$f(V \backslash U'') = f_2(U) \qquad U \subseteq S$$
$$f(S') = \min\{f_1(S), f_2(S)\}$$

Claim 10 *f is crossing submodular on $\mathscr{C}$.*

Exercise 11 *Prove Claim 10.*

Now define $G = (V, A)$ where $A = \{(s'', s') | s \in S\}$, as shown in Figure 1. The submodular flow polyhedron is

$$x(\delta^-(Z)) - x(\delta^+(Z)) \le f(Z) \qquad Z \in \mathscr{C}$$

If $Z = U'$ where $U \subseteq S$, then we get $x(U) \le f_1(U)$. And if $Z = V \backslash U''$, as shown in Figure 2, then we get $x(U) \le f_2(U), U \subseteq S$. Thus, we recover the polymatroid intersection constraints. Since the submodular flow constraint inequalities are box-TDI, it implies that the polymatroid intersection constraints are also box-TDI.

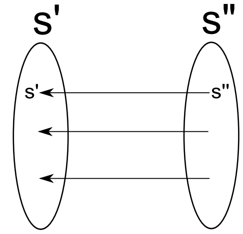

Figure 1: A Directed Graph Defined on S

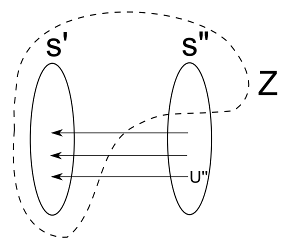

Figure 2: $Z = V \backslash U''$

2.3 Nash-Williams Graph Orientation Theorem

We stated the following theorem of Nas-Williams in the previous lecture.

Theorem 12 *If G is $2k$-edge-connected then it has an orientation D such that D is k-arc-connected.*

Frank(1980) [2] showed that Theorem 12 can be derived from submodular flows. Consider an arbitrary orientation D of G. Now if D is k-arc-connected we are done. Otherwise we consider the problem of reversing the orientation of some arcs of D such that the resulting graph is k-arc-connected. We set it up as follows.

Let $D = (V, A)$, define a variable $x(a), a \in A$ where $x(a) = 1$ if we reverse the orientation of a. Otherwise $x(a) = 0$. For a set $U \subset V$ we want k arcs coming in after applying the switch of orientation, i.e., we want

$$x(\delta^-(u)) - x(\delta^+(u)) \leq |\delta^-(u)| - k \qquad \forall\, \emptyset \subset U \subset V$$

Note that $\mathscr{C} = \{U \mid \emptyset \subset U \subset V\}$ is a crossing family and $f(U) = |\delta_D^-(u)| - k$ is crossing submodular. Hence by Edmonds-Giles theorem, the polyhedron determined by the inequalities

$$\begin{aligned} x(\delta^-(u)) - x(\delta^+(u)) &\leq |\delta^-(u)| - k \qquad \emptyset \subset U \subset V \\ x(a) &\in [0,1] \qquad a \in A \end{aligned}$$

is an integer polyhedron. Moreover, the polyhedron is non-empty since $x(a) = 1/2, \forall a \in A$ satisfies all the constraints. To see this, let $\emptyset \subset U \subset V$, and let $h = \delta_D^-(U)$ and $\ell = \delta_D^+(U)$, then we have $h + \ell \geq 2k$ since G is $2k$-edge-connected. Then by setting $x(a) = 1/2, \forall a \in A$, for U we need

$$\frac{h}{2} - \frac{l}{2} \leq h - k \Rightarrow \frac{h+l}{2} \leq k$$

which is true. Thus there is an integer vector x in the polyhedron for D if G is $2k$-edge-connected. By reversing the arcs $A' = \{a | x(a) = 1\}$ in D we obtain a k-arc-connected orientation of G.

2.4 Lucchesi-Younger theorem

Theorem 13 (Lucchesi-Younger) *In any weakly-connected digraph, The size of the minimum cardinality dijoin equals the maximum number of disjoint directed cuts.*

Proof: Let $D = (V, A)$ be a directed graph and let $\mathscr{C} = \{U \mid \emptyset \subset U \subset V, |\delta^+(U)| = 0\}$, i.e., $U \in \mathscr{C}$ iff U induces a directed cut. We had seen that $\mathscr{C}$ is a crossing family. Let $f : \mathscr{C} \to \mathbb{R}$ be $f(U) = -1, \forall U \in \mathscr{C}$, clearly f is crossing submodular. Then by Edmonds-Giles theorem the following set of inequalities is TDI.

$$\begin{aligned} x(\delta^-(U)) - x(\delta^+(U)) &\leq -1 \qquad U \in \mathscr{C} \\ x &\leq 0 \end{aligned}$$

We note that $\delta^+(U) = \emptyset$ for each $U \in \mathscr{C}$. We can rewrite the above polyhedron as the one below by replacing $-x$ by x.

$$\begin{aligned} x(\delta^-(U)) &\geq 1 \qquad U \in \mathscr{C} \\ x &\geq 0 \end{aligned}$$

Note that the above is a "natural" LP relaxation for finding a set of arcs that cover all directed cuts. The above polyhedron is integral, and hence

$$\begin{aligned} \min \sum_{a \in A} x(a) & \\ x(\delta^-(U)) &\geq 1 \qquad U \in \mathscr{C} \\ x &\geq 0 \end{aligned}$$

gives the size of a minimum cardinality dijoin.

Consider the dual

$$\begin{aligned} & \max \sum_{U \in \mathscr{C}} y(U) \\ \text{s.t.} \quad & \sum_{U : a \in \delta^-(U), U \in \mathscr{C}} y(U) \leq 1 \qquad a \in A \\ & y \geq 0 \end{aligned}$$

The dual is an integer polyhedron since the primal inequality system is TDI and the objective function is an integer vector. It is easy to see that the optimum value of the dual is a maximum packing of arc-disjoint directed cuts. Therefore by strong duality we obtain the Lucchesi-Younger theorem. $\square$

References

[1] Lex Schrijver, "Combinatorial Optimization: Polyhedra and Efficiency, Vol. B", *Springer-Verlag*, 2003.

[2] Lecture notes from Michael Goemans's class on Combinatorial Optimization, http://www-math.mit.edu/~goemans/18997-CO/co-lec18.ps, 2004.