We describe two well known theorems in combinatorial optimization. We prove the theorems using submodular flows later.

1 Graph Orientation

Definition 1 *Let $G = (V, E)$ be an undirected graph. For $u, v \in V$, we denote by $\lambda_G(u, v)$ the edge-connectivity between u and v in G, that is, the maximum number of edge-disjoint paths between u and v. Similarly for a directed graph $D = (V, A)$, $\lambda_D(u, v)$ is the maximum number of arc-disjoint paths from u to v.*

Note that for an undirected graph G, $\lambda_G(u, v) = \lambda_G(v, u)$ but it may not be the case that $\lambda_D(u, v) = \lambda_D(v, u)$ in a directed graph D.

Definition 2 *G is k-edge-connected if $\lambda_G(u, v) \geq k$ $\forall u, v \in V$. Similarly, D is k-arc-connected if $\lambda_D(u, v) \geq k$ $\forall u, v \in V$.*

Proposition 1 *G is k-edge-connected iff $|\delta(S)| \geq k$ $\forall S \subset V$. D is k-arc-connected iff $|\delta^+(S)| \geq k$ $\forall S \subset V$.*

Proof: By Menger's theorem. □

Definition 3 *$D = (V, A)$ is an orientation of $G = (V, E)$ if D is obtained from G by orienting each edge $uv \in E$ as an arc (u, v) or (v, u).*

Theorem 2 (Robbins 1939) *G can be oriented to obtain a strongly-connected directed graph iff G is 2-edge-connected.*

Proof: "⇒" Suppose $D = (V, A)$ is a strongly connected graph obtained as an orientation of $G = (V, E)$. Then, since $\forall S \subset V$, $|\delta^+_D(S)| \geq 1$ and $|\delta^-_D(S)| \geq 1$, we have $|\delta_G(S)| \geq 2$. Therefore, G is 2-edge-connected.

"⇐" Let G be a 2-edge-connected graph. Then G has an ear-decomposition. In other words, G is either a cycle C or G is obtained from a 2-edge-connected graph G' by adding an ear P (a path) connecting two not-necessarily distinct vertices $u, v \in V$.

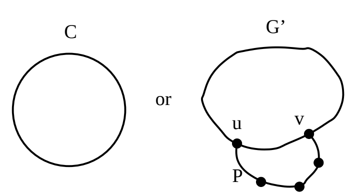

Figure 1: G is either a cycle C or is G' plus an ear P.

If $G = C$, orient it to obtain a directed cycle which is strongly-connected. Otherwise, inductively, G' has an orientation that is strongly-connected. Extend the orientation of G' to G by orienting

P from u to v (or v to u). It is easy to check that this orientation results in strongly-connected graph. $\square$

An alternative proof is as follows. Do a depth-first-search (DFS) of G starting at some node r. One obtains a DFS tree T. Orient all edges of T away from r to obtain an arborescence. Every other edge is a back-edge, that is if $uv \in E(G) \backslash E(T)$, then, either u is the ancestor of v in T or v is an ancestor of u in T. Orient uv from the descendant to the ancestor. We leave it as an exercise to argue that this is a strongly-connected orientation of G iff G is 2-edge-connected. Note that this is an easy linear time algorithm to obtain the orientation.

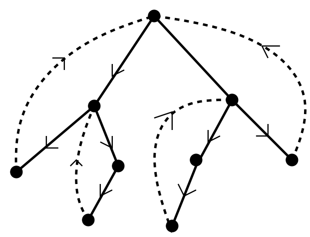

Figure 2: Orientation of a 2-edge-connected graph via a DFS tree.

Nash-Williams proved the following non-trivial extension.

Theorem 3 (Nash-Williams) *If G is $2k$-edge-connected, then it has an orientation that is k-arc-connected.*

In fact, he proved the following deep result, of which the above is a corollary.

Theorem 4 (Nash-Williams) *G has an orientation D in which $\lambda_D(u,v) \geq \lfloor \lambda_G(u,v)/2 \rfloor$ for all $u, v \in V$.*

The proof of the above theorem is difficult - see [1]. We will prove the easier version using submodular flows later.

2 Directed Cuts and Lucchesi-Younger Theorem

Definition 4 *Let $D = (V, A)$ be a directed graph. We say that $C \subset A$ is a directed cut if $\exists S \subset V$ such that $\delta^+(S) = \emptyset$ and $C = \delta^-(S)$.*

If D has a directed cut then D is *not* strongly-connected.

Definition 5 *A* dijoin *(also called a directed cut cover) in $D = (V, A)$ is a set of arcs in A that intersect each directed cut of D.*

It is not difficult to see that the following are equivalent:

- $B \subseteq A$ is a dijoin.

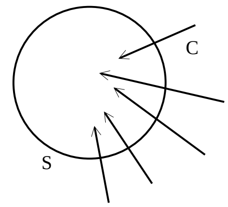

Figure 3: A directed cut $C = \delta^-(S)$.

- shrinking each arc in B results in a strongly-connected graph.
- adding all reverse arcs of B to D results in a strongly-connected graph.

Given $B \subseteq A$, it is therefore, easy to check if B is a dijoin; simply add the reverse arcs of B to D and check if the resulting digraph is strongly connected or not.

Definition 6 *A digraph D is weakly-connected if the underlying undirected graph is connected.*

Theorem 5 (Lucchesi-Younger) *Let $D = (V, A)$ be a weakly-connected digraph. Then the minimum size of a dijoin is equal to the maximum number of disjoint directed cuts.*

A dijoin intersects every directed cut so its size is at least the the maximum number of disjoint directe cuts. The above theorem is yet another example of a min-max result. We will prove this later using submodular flows. One can derive easily a weighted version of the theorem.

Corollary 6 *Let $D = (V, A)$ be a digraph with $\ell : A \to \mathbb{Z}_+$. Then the minimum length of a dijoin is equal to the maximum number of directed cuts such that each arc a is in at most $\ell(a)$ of them (in other words a maximum packing of directed cuts in ℓ).*

Proof: If $\ell(a) = 0$, contract it. Otherwise replace a by a path of length $\ell(a)$. Now apply the Lucchesi-Younger theorem to the modified graph. $\Box$

As one expects, a min-max result also leads to a polynomial time algorithm to compute a minimum weight dijoin and a maximum packing of directed cuts.

Woodall conjectured the following, which is still open. Some special cases have been solved [1].

Conjecture 1 (Woodall) *For every directed graph, the minimum size of a directed cut equals to the maximum number of disjoint dijoins.*

We describe an implication of Lucchesi-Younger theorem.

Definition 7 *Given a directed graph $D = (V, A)$, $A' \subseteq A$ is called a feedback arc set if $D[A \setminus A']$ is acyclic, that is, A' intersects each directed cycle of D.*

Computing a minimum cardinality feedback arc set is NP-hard. Now suppose D is a plane directed graph (i.e., a directed graph that is embedded in the plane). Then one defines its dual graph D^* as follows. For each arc (w, x) of D, we have a dual arc $(y, z) \in D^*$ that crosses (w, x) from "left" to "right". See example below.

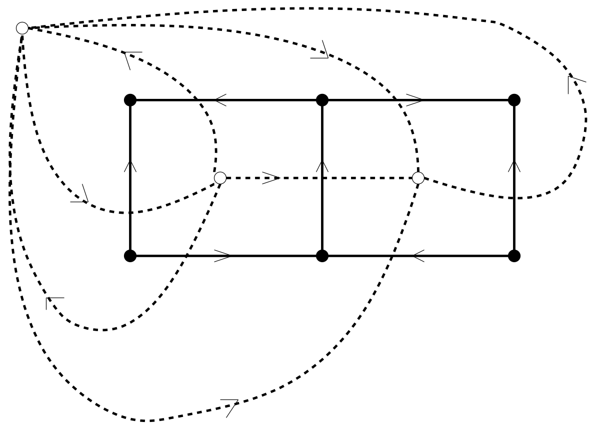

Figure 4: A planar digraph and its dual.

Proposition 7 *The directed cycles of D correspond to directed cuts in D^* and vice versa.*

Thus, a feedback arc set of D corresponds to a dijoin in D^*. Via Lucchesi-Younger theorem, we have the following corollary.

Corollary 8 *For a planar directed graph, the minimum size of a feedback arc set is equal to the maximum number of arc-disjoint directed cycles.*

Using the algorithm to compute a minimum weight dijoin, we can compute a minimum weight feedback arc set of a planar digraph in polynomial time.

3 Polymatroid Intersection

Recall the definition of total dual integrality of a system of inequalities.

Definition 8 *A rational system of inequalities $Ax \leq b$ is TDI if for all integral c,* $\min\{yb \mid y \geq 0, yA = c\}$ *is attained by an integral vector y^* whenever the optimum exists and is finite.*

Definition 9 *A rational system of inequalities $Ax \leq b$ is box-TDI if the system $d \leq x \leq c$, $Ax \leq b$ is TDI for each $d, c \in \mathcal{R}^n$.*

In particular, we have the following. If $Ax \leq b$ is box-TDI, then the polyhedron $\{x \mid Ax \leq b, d \leq \ell \leq u\}$ is an integer polyhedron whenever b, ℓ, u are integer vectors.

Recall that if $f : 2^S \rightarrow \mathcal{R}$ is a submodular function, EP_f is the extended polymatroid defined as

$$\{x \in \mathcal{R}^S | x(U) \leq f(U), U \subseteq S\}$$

We showed that the system of inequalities $x(U) \leq f(U), U \subseteq S$ is TDI. In fact, one can show that the system is also box-TDI. Polymatroids generalize matroids. One can also consider polymatroid intersection which generalizes matroid intersection.

Let f_1, f_2 be two submodular functions on S. Then the polyhedron $EP_{f_1} \cap EP_{f_2}$ described by

$$\begin{aligned} x(U) &\le f_1(U) \qquad U \subseteq S \\ x(U) &\le f_2(U) \qquad U \subseteq S \end{aligned}$$

is an integer polyhedron whenever f_1 and f_2 are integer valued. We sketch a proof of the following theorem.

Theorem 9 (Edmonds) *Let f_1, f_2 be two submodular set functions on the ground set S. The system of inequalities*

$$\begin{aligned} x(U) &\le f_1(U) \qquad U \subseteq S \\ x(U) &\le f_2(U) \qquad U \subseteq S \end{aligned}$$

is box-TDI.

Proof: (Sketch) The proof is similar to that of matroid intersection. Consider primal-dual pair below

$$\begin{aligned} &\max wx \\ x(U) &\le f_1(U) \qquad U \subseteq S \\ x(U) &\le f_2(U) \qquad U \subseteq S \\ &\ell \le x \le u \end{aligned}$$

$$\begin{gathered} \min \textstyle\sum_{U \subseteq S} (f_1(U) y_1(U) + f_2(U) y_2(U)) + \sum_{a \in S} u(a) z_1(a) - \sum_{a \in S} \ell(a) z_2(a) \\ \textstyle\sum_{a \in U} (y_1(U) + y_2(U)) + z_1(a) - z_2(a) = w(a), a \in S \\ y \ge 0, z_1, z_2 \ge 0 \end{gathered}$$

Claim 10 *There exists an optimal dual solution such that $\mathcal{F}_1 = \{U \mid y_1(U) > 0\}$ and $\mathcal{F}_2 = \{U \mid y_2(U) > 0\}$ are chains.*

The proof of the above claim is similar to that in matroid intersection. Consider $\mathcal{F}_1 = \{U \mid y_1(U) > 0\}$. If it is not a chain, there exist $A, B \in \mathcal{F}_1$ such that $A \not\subset B$ and $B \not\subset A$. We change y_1 by adding ϵ to $y_1(A \cup B)$ and $y_1(A \cap B)$ and subtracting ϵ from $y_1(A)$ and $y_1(B)$. One observes that the feasibility of the solution is maintained and that the objective function can only decrease since f_1 is submodular. Thus, we can uncross repeatedly to ensure that $\mathcal{F}_1$ is a chain, similarly $\mathcal{F}_2$.

Let y_1, y_2, z_1, z_2 be an optimal dual solution such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are chains. Consider $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ and the $S \times \mathcal{F}$ incidence matrix M. As we saw earlier in the proof for matroid intersection, M is TUM. We then have y_1, y_2, z_1, z_2 are determined by a system $[\, y_1 \quad y_2 \quad z_1 \quad z_2 \,][\, M \quad I \quad -I \,] = w$, where w is integer and M is TUM. Since $[\, M \quad I \quad -I \,]$ is TUM, there exists integer optimum solution. □

Note that, one can separate over $EP_{f_1} \cap EP_{f_2}$ via submodular function minimization and hence one can optimize $EP_{f_1} \cap EP_{f_2}$ in polynomial time via the ellipsoid method. Strongly polynomial time algorithm can also be derived. See [1] for details.

4 Submodularity on Restricted Families of Sets

So far we have seen submodular functions on a ground set S. That is $f : 2^S \to R$ and $\forall A, B \subseteq S$,

$$f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$$

In several applications, one needs to work with restricted families of subsets. Given a finite set S, a family of sets $\mathcal{C} \subseteq 2^S$ is

- a *lattice family* if $\forall A, B \in \mathcal{C}$, $A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$.
- an *intersecting family* if $\forall A, B \in \mathcal{C}$ and $A \cap B \neq \emptyset$, we have $A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$.
- a *crossing family* if $A, B \in \mathcal{C}$ and $A \cap B \neq \emptyset$ and $A \cup B \neq S$, we have $A \cap B \in \mathcal{C}$ and $A \cup B \in \mathcal{C}$.

For each of the above families, a function f is submodular on the family if

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$$

whenever $A \cap B$, $A \cup B$ are guaranteed to be in family for A, B. Function f is called intersection submodular and crossing submodular if $\mathcal{C}$ is intersecting and crossing family respectively.

We give some examples of interesting families that arise from directed graphs. Let $D = (V, A)$ be a directed graph.

Example 1 *$\mathcal{C} = 2^V \setminus \{\emptyset, V\}$ is a crossing family.*

Example 2 *Fix $s, t \in V$, $\mathcal{C} = \{U \mid s \in U, t \notin U\}$ is lattice, intersecting, and crossing family.*

Example 3 *$\mathcal{C} = \{U \subset V \mid U$ induces a directed cut i.e. $\delta^+(U) = \emptyset$ and $\emptyset \subset U \subset V\}$ is a crossing family.*

For the above example, we sketch the proof that $\mathcal{C}$ is a crossing family. If $A, B \in \mathcal{C}$ and $A \cap B \neq \emptyset$ and $A \cup B \neq V$, then by submodularity of δ^+, $|\delta^+(A \cup B)| + |\delta^+(A \cap B)| \leq |\delta^+(A)| + |\delta^+(B)|$. Therefore we have $\delta^+(A \cup B) = \emptyset$ and $\delta^+(A \cap B) = \emptyset$ and more over $A \cap B$ and $A \cup B$ are non-empty. Hence they both belong to $\mathcal{C}$ as desired.

Various polyhedra associates with submodular functions and the above special families are known to be well-behaved.

For lattice families the system

$$x(U) \quad \leq \quad f(U), U \in \mathcal{C}$$

is box-TDI. Also, the following system is also box-TDI

$$\begin{aligned} x(U) &\leq f_1(U), U \in \mathcal{C}_1 \\ x(U) &\leq f_2(U), U \in \mathcal{C}_2 \end{aligned}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are lattice families and f_1 and f_2 are submodular on $\mathcal{C}_\infty$ and $\mathcal{C}_2$ respectively. The above facts also hold for intersecting families and intersecting submodular functions.

For crossing family $\mathcal{C}$, the system

$$x(U) \leq f(U)$$

is not necessarily TDI. However, the system

$$\begin{aligned} x(U) &\leq f(U), U \in \mathcal{C} \\ x(S) &= k \end{aligned}$$

where $k \in R$ is box-TDI. Also, the system

$$\begin{aligned} x(U) &\leq f_1(U), U \in \mathcal{C}_1 \\ x(U) &\leq f_2(U), U \in \mathcal{C}_2 \\ x(S) &= k \end{aligned}$$

is box-TDI for crossing families $\mathcal{C}_1$ and $\mathcal{C}_2$ with f_1 and f_2 crossing supermodular on $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively.

Although the polyhedra are well-behaved, the separation problem for them is not easy since one needs to solve submodular function minimization over a restricted family $\mathcal{C}$. It does not suffice to have a value oracle for f on sets in $\mathcal{C}$; one needs additional information on the representation of $\mathcal{C}$. We refer the reader to [1] for more details.

References

[1] A. Schrijver. *Combinatorial Optimization.* Springer-Verlag Berlin Heidelberg, 2003.

[2] Lecture notes from Michel Goemans's class on Combinatorial Optimization. http://www-math.mit.edu/ goemans/18997-CO/co-lec18.ps