1 Total Dual Integrality

Recall that if $A$ is TUM and $b,c$ are integral vectors, then $\max\{cx : Ax \leq b\}$ and $\min\{yb : y \geq 0, yA = c\}$ are attained by integral vectors $x$ and $y$ whenever the optima exist and are finite. This gives rise to a variety of min-max results, for example we derived König’s theorem on bipartite graphs. There are many examples where we have integral polyhedra defined by a system $Ax \leq b$ but $A$ is not TUM; the polyhedron is integral only for some specific $b$. We may still ask for the following. Given any $c$, consider the maximization problem $\max\{cx : Ax \leq b\}$; is it the case that the dual minimization problem $\min\{yb : y \geq 0, yA = c\}$ has an integral optimal solution (whenever a finite optimum exists)?

This motivates the following definition:

**Definition 1** A rational system of inequalities $Ax \leq b$ is totally dual integral (TDI) if, for all integral $c$, $\min\{yb : y \geq 0, yA = c\}$ is attained by an integral vector $y^*$ whenever the optimum exists and is finite.

**Remark 2** If $A$ is TUM, $Ax \leq b$ is TDI for all $b$.

This definition was introduced by Edmonds and Giles\[2\] to set up the following theorem:

**Theorem 3** If $Ax \leq b$ is TDI and $b$ is integral, then $\{x : Ax \leq b\}$ is an integral polyhedron.

This is useful because $Ax \leq b$ may be TDI even if $A$ is not TUM; in other words, this is a weaker sufficient condition for integrality of $\{x : Ax \leq b\}$ and moreover guarantees that the dual is integral whenever the primal objective vector is integral.

**Proof Sketch.** Let $P = \{x : Ax \leq b\}$. Recall that we had previously shown that the following are equivalent:

(i) $P$ is integral.

(ii) Every face of $P$ contains an integer vector.

(iii) Every minimal face of $P$ contains an integer vector.

(iv) $\max\{cx : x \in P\}$ is achieved by an integer vector whenever the optimum is finite.

Edmonds and Giles proved two more equivalent conditions:

(v) Every rational supporting hyperplane of $P$ contains an integer vector.

(vi) If $c$ is integral, then $\max\{cx : x \in P\}$ is an integer whenever the optimum exists and is finite.
Condition (vi) implies the theorem as follows. If \( Ax \leq b \) is TDI and \( b \) is integral, \( \max \{ cx : x \in P \} \) is an integer for all integral \( c \) whenever it is finite; this is because the dual optimum is achieved by an integral vector \( y^* \) (TDI property) and the objective function \( by^* \) is integral because \( b \) is integral. This implies that \( P \) is integral.

There’s an important subtlety to the definition of total dual integrality: being TDI is a property of a system of inequalities, not a property of the corresponding polyhedron.

We will illustrate this with an example from [3]. Consider the system \( Ax \leq b \) drawn above on the left. If we take the cost vector \( c \) to be \((1,1)\), then the primal has an optimum at \((2,2)\) with value 4. The tight constraints at this vertex have normal vectors \((2,1)\) and \((1,2)\) (these are rows of \( A \)). Therefore, in order for the dual \( yA = c \) to have an integer solution, we must be able to express \((1,1)\) as an integer combination of \((2,1)\) and \((1,2)\). Since this is impossible, \( Ax \leq b \) is not TDI.

However, suppose we add more constraints to obtain the system \( A'x \leq b' \) drawn above on the right. Note that this system corresponds to the same polyhedron as \( Ax \leq b \). However, now we have an additional normal vector at \((2,2)\) – namely, \((1,1)\). Thus \((1,1)\) is now an integer combination of the normal vectors at \((2,2)\). The system \( A'x \leq b' \) is in fact TDI, even though it corresponds to the same polytope as the (non-TDI) system \( Ax \leq b \).

The example demonstrates a necessary for a system to be TDI. We explain this in the general context. Consider the problem \( \max \{ cx : Ax \leq b \} \) with \( c \) integral, and assume it has a finite optimum \( \beta \). Then it is achieved by some vector \( x^* \) in the face \( F \) defined by the intersection of \( \{ x : Ax \leq b \} \) with the hyperplane \( cx = \beta \). For simplicity assume that the face \( F \) is an extreme point/vertex of the polyhedron and let \( A'x^* = b' \) be the set of all inequalities in \( Ax \leq b \) that are tight at \( x^* \). The dual is \( \min \{ yb : y \geq 0, yA = c \} \). By LP duality theory, any dual optimum solution \( y \) corresponds to \( c \) being expressed a non-negative combination of the row vectors of \( A' \), in other words \( c \) is in the cone of the row vectors of \( A' \). If \( Ax \leq b \) is TDI then we ask for an integral dual optimum solution; this requires that there is an integer solution to \( yA' = c, y \geq 0 \). This motivates
the following definition.

**Definition 4** A set \{a_1, \ldots, a_k\} of vectors in \(\mathbb{R}^n\) is a Hilbert basis if every integral vector \(x \in \text{Cone}(\{a_1, \ldots, a_k\})\) can be written as \(x = \sum_{i=1}^{k} \mu_i a_i, \mu_i \geq 0, \mu_i \in \mathbb{Z}\) (that is, \(x\) is a non-negative integer combination of \(a_1, \ldots, a_k\)). If the \(a_i\) are themselves integral, we call \{a_1, \ldots, a_k\} an integral Hilbert basis.

The following theorem is not difficult to prove with the background that we have developed.

**Theorem 5** The rational system \(Ax \leq b\) is TDI if and only if the following property is true for each face \(F\) of \(P\); let \(A'x = b'\) be the set of all inequalities in \(Ax \leq b\) that are tight/active at \(F\), then the rows vectors of \(A'\) form a Hilbert basis.

**Corollary 6** If the system \(Ax \leq b, \alpha x \leq \beta\) is TDI then \(Ax \leq b, \alpha x = \beta\) is also TDI.

The example above raises the question of whether one can take any rational system \(Ax \leq b\) and make it TDI by adding sufficiently many redundant inequalities. Indeed that is possible, and is based on the following theorem.

**Theorem 7** Every rational polyhedral cone has a finite integral Hilbert basis.

**Theorem 8 (Giles-Pulleyblank)** Any rational polyhedron \(P\) has a representation \(Ax \leq b\) such that

(i) \(P = \{x : Ax \leq b\}\),

(ii) \(A\) is integral, and

(iii) \(Ax \leq b\) is TDI.

Moreover, \(b\) is integral if and only if \(P\) is integral.

2 The Cunningham-Marsh Theorem

Suppose we have a graph \(G = (V,E)\). Let \(P_{\text{odd}}(V)\) denote the family of all odd subsets of \(V\) with size at least 3. Recall that in our study of matchings, we have examined three different systems of inequalities.

\[
P_1: \begin{align*}
x(\delta(v)) &= 1 & \forall v \in V \\
x(\delta(U)) &\geq 1 & U \in P_{\text{odd}}(V) \\
x &\geq 0
\end{align*}
\]

\[
P_2: \begin{align*}
x(\delta(v)) &\leq 1 & \forall v \in V \\
x(E[U]) &\leq \left\lfloor \frac{1}{2} |U| \right\rfloor & U \in P_{\text{odd}}(V) \\
x &\geq 0
\end{align*}
\]

\[
P_3: \begin{align*}
x(\delta(v)) &= 1 & \forall v \in V \\
x(E[U]) &\leq \left\lfloor \frac{1}{2} |U| \right\rfloor & U \in P_{\text{odd}}(V) \\
x &\geq 0
\end{align*}
\]
Here $P_2$ determines the matching polytope for $G$, while $P_1$ and $P_3$ determine the perfect matching polytope.

It is not hard to see that $P_1$ is not TDI. Consider $K_4$ with $w(e) = 1$ for every edge $e$. In this case, the unique optimal dual solution is $y_v = \frac{1}{2}$ for each vertex $v$.

On the other hand, $P_2$ and $P_3$ are TDI; this was proven by Cunningham and Marsh\[1\]. Consider the primal maximization and dual minimization problems for $P_2$ below:

maximize $wx$ subject to
\[
x(\delta(v)) \leq 1 \quad \forall v \in V \\
x(E[U]) \leq \lfloor \frac{1}{2} |U| \rfloor \quad \forall U \in P_{odd}(V) \\
x \geq 0
\]

minimize $\sum_{v \in V} y_v + \sum_{U \in P_{odd}(V)} z_U \cdot \left\lfloor \frac{1}{2} |U| \right\rfloor$ subject to
\[
y_a + y_b + \sum_{U \in P_{odd}(V)} z_U \geq w(ab) \quad \forall ab \in E \\
y \geq 0, z \geq 0
\]

By integrality of the matching polytope, the maximum value of the primal is the maximum weight of a matching under $w$; by duality, this equals the minimum value of the dual. The Cunningham-Marsh Theorem tells us that this minimum value is achieved by integral dual vectors $y^*, z^*$ with the additional condition that the sets $\{ U : z_U^* > 0 \}$ form a laminar family.

**Theorem 9 (Cunningham-Marsh)** The system $P_2$ is TDI (as is $P_3$). More precisely, for every integral $w$, there exist integral vectors $y$ and $z$ that are dual feasible such that $\{ U : z_U > 0 \}$ is laminar and
\[
\sum_{v \in V} y_v + \sum_{U \in P_{odd}(V)} z_U \cdot \left\lfloor \frac{1}{2} |U| \right\rfloor = \nu(w)
\]

where $\nu(w)$ is the maximum weight of a matching under $w$.

**Exercise 10** Show that the Tutte-Berge Formula can be derived from the Cunningham-Marsh Theorem.

Cunningham and Marsh originally proved this theorem algorithmically, but we present a different proof from \[1\]; the proof relies on the fact that $P_2$ is the matching polytope. A different proof is given in \[1\] that does not assume this and in fact derives that $P_2$ is the matching polytope as a consequence.

**Proof:** We will use induction on $|E| + w(E)$ (which is legal because $w$ is integral). Note that if $w(e) \leq 0$ for some edge $e$, we may discard it; hence we may assume that $w(e) \geq 1$ for all $e \in E$.

**Case I: Some vertex $v$ belongs to every maximum-weight matching under $w$.**

Define $w' : E \to \mathbb{Z}^+$ by
\[
w'(e) = w(e) - 1 \quad \text{if} \ e \in \delta(v) \\
w'(e) = w(e) \quad \text{if} \ e \notin \delta(v)
\]

Now induct on $w'$. Let $y', z'$ be an integral optimal dual solution with respect to $w'$ such that $\{ U : z_U^* > 0 \}$ is laminar; the value of this solution is $\nu(w')$. Because $v$ appears in every maximum-weight matching under $w$, $\nu(w') \leq \nu(w) - 1$; by definition of $w'$, $\nu(w') \geq \nu(w) - 1$. Thus $\nu(w') = \nu(w) - 1$.
Let $y^*$ agree with $y'$ everywhere except $v$, and let $y^*_v = y'_v + 1$. Let $z^* = z'$. Now $y^*, z^*$ is a dual feasible solution with respect to $w$, the solution is optimal since it has weight $\nu(w') + 1 = \nu(w)$, and $\{U : z^*_U > 0\}$ is laminar since $z^* = z'$.

**Case II:** No vertex belongs to every maximum-weight matching under $w$.

Let $y, z$ be a *fractional* optimal dual solution. Observe that $y = 0$, since $y_v > 0$ for some vertex $v$, together with complementary slackness, would imply that every optimal primal solution covers $v$, i.e. $v$ belongs to every maximum-weight matching under $w$. Among all optimal dual solutions $y, z$ (with $y = 0$) choose the one that maximizes $\sum_{U \in P_{\text{odd}}(V)} z_U \left( \frac{1}{2} |U| \right)^2$. To complete the proof, we just need to show that $z$ is integral and $\{U : z_U > 0\}$ is laminar.

Suppose $\{U : z_U > 0\}$ is not laminar; choose $W, X \in P_{\text{odd}}(V)$ with $z_W > 0, z_X > 0$, and $W \cap X \neq \emptyset$. We claim that $|W \cap X|$ is odd. Choose $v \in W \cap X$, and let $M$ be a maximum-weight matching under $w$ that misses $v$. Since $z_W > 0$, by complementary slackness, $M$ contains $\left( \frac{1}{2} |W| \right)$ edges inside $W$; thus $v$ is the *only* vertex in $W$ missed by $M$. Similarly, $v$ is the only vertex in $X$ missed by $M$. Thus $M$ covers $W \cap X - \{v\}$ using only edges inside $W \cap X - \{v\}$, so $|W \cap X - \{v\}|$ is even, and so $|W \cap X|$ is odd. Let $\epsilon$ be the smaller of $z_W$ and $z_X$; form a new dual solution by decreasing $z_W$ and $z_X$ by $\epsilon$ and increasing $z_{W \cap X}$ and $z_{W \cup X}$ by $\epsilon$ (this is an uncrossing step).

We claim that this change maintains dual feasibility and optimality. Clearly $z_W$ and $z_X$ are still nonnegative. If an edge $e$ is contained in $W$ and $X$, then the sum in $e$’s dual constraint loses $2\epsilon$ from $z_W$ and $z_X$, but gains $2\epsilon$ from $z_{W \cap X}$ and $z_{W \cup X}$, and hence still holds. Likewise, if $e$ is contained in $W$ but not $X$ (or vice-versa), the sum loses $\epsilon$ from $z_W$ but gains $\epsilon$ from $z_{W \cap X}$. Thus these changes maintained dual feasibility and did not change the value of the solution, so we still have an optimal solution. However, we have increased $\sum_{U \in P_{\text{odd}}(V)} z_U \left( \frac{1}{2} |U| \right)^2$ (the reader should verify this), which contradicts the choice of $z$. Thus $\{U : z_U > 0\}$ is laminar.

Suppose instead that $z$ is not integral. Choose a maximal $U \in P_{\text{odd}}(V)$ such that $z_U$ is not an integer. Let $U_1, \ldots, U_k$ be maximal odd sets contained in $U$ such that each $z_{U_i} > 0$. (Note that we may have $k = 0$.) By laminarity, $U_1, \ldots, U_k$ are disjoint. Let $\alpha = z_U - \lfloor z_U \rfloor$. Form a new dual solution by decreasing $z_U$ by $\alpha$ and increasing each $z_{U_i}$ by $\alpha$.

We claim that the resulting solution is dual feasible. Clearly we still have $z_U \geq 0$, and no other dual variable was decreased. Thus we need only consider the edge constraints; moreover, the only constraints affected are those corresponding to edges contained within $U$. Let $e$ be an edge contained in $U$. If $e$ is contained in some $U_i$, then the sum in $e$’s constraint loses $\alpha$ from $z_U$ but gains $\alpha$ from $z_{U_i}$, so the sum does not change. On the other hand, suppose $e$ is not contained in any $U_i$. By maximality of $U$ and the $U_i$, $U$ is the only set in $P_{\text{odd}}$ containing $e$. Thus before we changed $z_U$ we had $z_U \geq w(e)$; because $w(e)$ is integral, we must still have $z_U \geq w(e)$. Hence our new solution is dual feasible.

Since the $U_i$ are disjoint, contained in $U$, and odd sets, $\left( \frac{1}{2} |U| \right) > \sum_{i=1}^k \left( \frac{1}{2} |U_i| \right)$. Thus our new solution has a smaller dual value than the old solution, which contradicts the optimality of $z$. It follows that $z$ was integral, which completes the proof.

To show that the system $P_3$ is TDI, we use Corollary [10] and the fact that system $P_2$ is TDI. □

**References**

