Spring 2010, CS 598CC: Topics in Combinatorial Optimization
Homework 3

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince me that you know the solution, as quickly as possible.

Problem 1 Given a graph $G = (V, E)$ let $\mathcal{I} = \{S \subseteq V \mid \text{there is a matching } M \text{ in } G \text{ that covers } S\}$. Prove that $(V, \mathcal{I})$ is a matroid.

Problem 2 Let $G = (V, \mathcal{E})$ be a hypergraph, that is each $e \in \mathcal{E}$ is a hyperedge, in other words $e \subseteq V$. We say that $X \subseteq \mathcal{E}$ a *forest-representable* if one can choose for each $e \in X$ two nodes in e such that the chosen pairs when viewed as edges form a forest on V. Prove that $(\mathcal{E}, \mathcal{I})$ is a matroid where $\mathcal{I} = \{X \subseteq \mathcal{E} \mid X \text{ is forest-representable}\}$. This is called the hypergraphic matroid.

Problem 3 Let A be a full-rank $n \times n$ matrix over the reals. Let R and C be the index sets of the rows and columns. Given $I \subset R$ show that there exists $J \subset C$ such that $|I| = |J|$ and both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank. Use matroid intersection.

Problem 4 We saw that matroid union can be algorithmically reduced to matroid intersection. One can derive the matroid intersection theorem from matroid union theorem. Givem two matroids $\mathcal{M}_1 = (S, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2)$ let $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2^*$ be the union of $\mathcal{M}_1$ and the dual of $\mathcal{M}_2$. Let B be a base of $\mathcal{M}$; it follows that B can be partitioned into J_1 and J_2 where J_1 is independent in $\mathcal{M}_1$ and J_2 is independent in $\mathcal{M}_2^*$. Extend J_2 (in an arbitrary way) to a base B_2 in $\mathcal{M}_2^*$. Show that $B \setminus B_2$ is maximum cardinality common independent set of M_1 and M_2.

Problem 5 Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$ of nodes, define the *density* of the induced graph $G[S]$ as $|E[S]|/|S|$ where $E[S]$ is the set of edges with both end points in S. The goal is to find the set of nodes that maximizes the density of the induced graph. Suppose we are given a number λ and wish to check if there is a set S with density at least λ. Use submodular function minimization to solve this problem.