Spring 2010, CS 598CC: Topics in Combinatorial Optimization
Homework 1
Due: 2/25/2010 in class

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince me that you know the solution, as quickly as possible.

Do as many problems as you can, I expect you do at least 3.

Problem 1 From Lecture 3. Give proofs for Exercise 4 and Theorem 9.

Problem 2 Let $G = (V, E)$ be a bipartite graph and let k be an integer. Let S_k be the set of characterstic vectors of the matchings in G of size at most k. Show that the following polytope is the convex hull of the vectors in S_k.

$$\begin{aligned} \sum_{e \in E} x(e) &\leq k \\ x(\delta(u)) &\leq 1 \qquad u \in V \\ x(e) &\geq 0 \qquad e \in E \end{aligned}$$

One option is to show that the constraint matrix of the above polytope is TUM. Another is to show that all basic feasible solutions are integral.

Problem 3 Flow is typically defined as a function on the edges but path-based flow definition and formulations are useful and necessary in various applications. We illustrate some relevant issues via this problem. Let $D = (V, A)$ be a directed graph with non-negative capacities $c : A \rightarrow R^+$. Given nodes s, t let $P_{s,t}$ denote the set of all simple paths between s and t.

- Write the maximum s-t flow problem as a linear programming problem with one variable for each path $p \in P_{s,t}$. Note that the primal can have an exponential (in $|V|$) number of variables. Write its dual.
- What is the separation problem for the dual? Show that there is a polynomial time algorithm for the separation problem for the dual. Via the Ellipsoid method, this implies that you can solve the dual to optimality.
- Suppose you have an optimum solution to the dual. Show via complementary slackness that you can restrict attention to only a polynomial number of paths in the primal and hence find an optimum solution to the primal by solving a linear program of size polynomial in the input.

- Via flow-decomposition observe that if the capacities are integral then the primal has an optimum integral solution. Does this imply that the polyhedron defined by the constraints is integral? If not, explain.
- Now consider the following problem. You want to find the maximum s-t flow when restricted to paths of length at most k where k is a given integer. Write this is a large linear program. Show that the separation oracle for the dual is polynomial time solvable.

Problem 4 Consider a bipartite graph $G = (V, E)$ with A, B as the bipartition of the vertex set B. Prove the following.

- A subset $X \subseteq A$ is matchable if there is matching that saturates X. Suppose X, Y are matchable and $|X| < |Y|$. Show that there is a $y \in Y \setminus X$ such that $X' = X \cup \{y\}$ is also matchable.
- For $X \subseteq A$, define $def(X)$ as $|X| - |N(X)|$ where $N(X)$ is the set of neighbors of X in B. Generalize Hall's theorem to show that the size of a maximum matching in G is equal to $|A| - k$ where $k = \max_{X \subseteq A} def(X)$. Note that $k \geq 0$ by taking $X = \emptyset$.
- Prove that $def : 2^A \to Z$ is supermodular, that is, for all $X, Y \subseteq A$,

$$def(X) + def(Y) \leq def(X \cap Y) + def(X \cup Y).$$

Problem 5 Consider the algorithm of Edmonds to find an M-augmenting path in a graph G via shrinking blossoms. Suppose G has no M-alternating X-X walk where X is the set of M-exposed nodes, then M is a maximum matching. In this case, can you algorithmically find a witness U in the Tutte-Berge formula to certify that M is a maximum matching? For this you need to work with a specific algorithm that tries to find a walk, for example the reduction to a directed path problem that we discussed in class/lecture notes (see also Schrijver's book). Note that one may find that M is a maximum matching in G after several recursive steps. Can you extend the witness U found in the recursive step to the original graph?