CS 598CSC: Approximation Algorithms	Lecture date: April 15, 2009
Instructor: Chandra Chekuri	Scribe: Qiang Ma

In computer science, when dealing with difficult problems involving graphs and their associated metrics, one technique we usually resort to is to solve the problem on a simple subcase, in particular, on tree. In this lecture, we study the Padded Decomposition of graphs and how to build their Tree metrics with $O(\log n)$ distortion.

1 Tree Metrics

Definition 1 Tree Metric - *A metric (V, d) is a Tree Metric if there exists a tree $T = (V, E_T)$ with edge lengths such that $d(u, v) = d_T(u, v)$.*

We can build tree metrics to approximate general graph metrics, with an $O(\log n)$ distortion, where n is the number of vertices in the graph, thus the following theorem:

Theorem 1 *Given an undirected edge weighted graph $G = (V, E)$, there is a randomized poly-time algorithm that produces an random edge weighted tree $T = (V_T, E_T)$ such that*

1. $V \subseteq V_T$;
2. $d_G(u, v) \leq d_T(u, v)$, $\forall u, v$;
3. $E(d_T(u, v)) \leq O(\log |V|) d_G(u, v)$, $\forall u, v$.

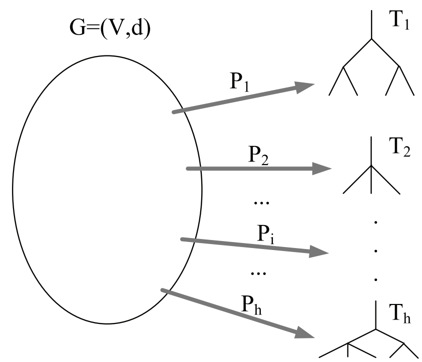

Figure 1: Tree Embedding.

The Problem that Theorem 1 indicates is as follows (refer to Figure 1 for illustration):

Problem 1 *Given a graph metric $G = (V, d)$, find a collection of trees $\mathscr{T} = \{T_1, T_2, \ldots, T_h\}$ and a distribution $\mu = \{P_1, P_2, \ldots, P_h\}$ on them such that*

1. $\sum_{i=1}^{h} P_i = 1$;
2. $d_{T_i}(u, v) \geq d_G(u, v)$, $\forall u, v$, $\forall i$;
3. $\sum_{i=1}^{h} P_i d_{T_i}(u, v) \leq O(\log n) d_G(u, v)$, $\forall u, v$.

A tree embedding algorithm is discussed in the next section to show the correctness of Theorem 1.

2 Padded Decomposition and Tree Embeddings

The following algorithm is developed by Bartal in [1] to solve the tree embedding problem. Note that $\Delta(G)$ denotes the diameter of graph G.

<u>TREE EMBEDDING ALGORITHM($G(V,d)$):</u>
Start with G.
Randomly decompose G into $G_1, G_2, \ldots, G_h$ s.t.
• $\Delta(G) \leq \frac{\Delta(G)}{2}$, $\forall 1 \leq i \leq h$;
• $Pr[(u,v) \text{ is cut}] \leq \frac{\alpha \cdot d_G(u,v)}{\Delta}$.
Recursively construct "rooted" trees $T_1, T_2, \ldots, T_h$ for $G_1, G_2, \ldots, G_h$, respectively.
Create tree T for G by (see Figure 2)
• adding root r;
• connecting $T_1, T_2, \ldots, T_h$ to r with edge weighted $\Delta(G)$.

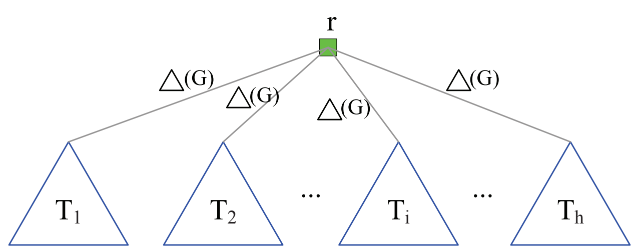

Figure 2: Create tree T from $T_1, T_2, \ldots, T_h$.

The above TREE EMBEDDING ALGORITHM is a high-level algorithm, we first suppose there exists a graph decomposition procedure to randomly decompose G into $G_1, G_2, \ldots, G_h$ as in the second step, we then have the following two Propositions that can be easily observed.

Proposition 2 *T dominates G.*

Proposition 3 *$\Delta(T) \leq 4\Delta(G)$.*

We subsequently have the following lemma:

Lemma 4 *$E(d_T(u,v)) = O(\alpha \log \Delta(G)) d_G(u,v)$.*

Proof of Lemma 4. Let A_i denote the event that (u,v) is cut in level i, and let $\bar{A}_i$ denote the event that (u,v) is not cut in level i.

$$
\begin{aligned}
E[d_T(u,v)] &\leq 4 \cdot \Delta(G) \cdot Pr[A_1] \\
&\quad +4 \cdot \frac{\Delta(G)}{2} \cdot Pr[A_2 | \bar{A}_1] \\
&\quad + \ldots + 4 \cdot \frac{\Delta(G)}{2^i} \cdot Pr[A_i | \cap_{j=1}^{i-1} \bar{A}_j] + \ldots \\
&= 4 \cdot \Delta(G) \cdot \frac{\alpha \cdot d_G(u,v)}{\Delta(G)} \\
&\quad +4 \cdot \frac{\Delta(G)}{2} \cdot \frac{\alpha \cdot d_G(u,v)}{\Delta(G)/2} \\
&\quad + \ldots + 4 \cdot \frac{\Delta(G)}{2^i} \cdot \frac{\alpha \cdot d_G(u,v)}{\Delta(G)/2^i} + \ldots \\
&\leq 4\alpha \log \Delta(G) \cdot d_G(u,v) \\
&= O(\alpha \log \Delta(G)) \cdot d_G(u,v)
\end{aligned}
$$

□

Note that now $E[d_T(u,v)]$ is dependent on $\log \Delta(G)$, and this can actually be replaced with $\log n$ if the CKR GRAPH DECOMPOSITION procedure is incorporated into the TREE EMBEDDING ALGORITHM, so that correctness of Theorem 1 can be proved. The CKR GRAPH DECOMPOSITION procedure is introduced in next subsection.

2.1 CKR Graph Decomposition Procedure

```
CKR GRAPH DECOMPOSITION(G(V,d), δ):
Pick θ at random from [δ/4, δ/2].
Pick a random permutation σ on V.
For i = 1 to n do
   G_i = B(V_σ(i), θ) \ ⋃_{j<i} B(V_σ(j), θ)
```

Note that $B(u,\theta)$ denotes the ball that is centered at u and with radius as θ, and we say $B(u,\theta)$ is cut if any vertex in $B(u,\theta)$ is separated from u.

Theorem 5 *CKR decomposition decomposes G into $G_1, G_2, \ldots, G_n$ such that:*

1. $\Delta(G_i) \le \delta$;
2. $\forall u$, $Pr[B(u,\varrho)$ *is cut* $] \le \frac{c\varrho}{\delta}\log n$, *where c is a constant.*

It is easy to see the following is an immediate Corollary of Theorem 5:

Corollary 6 $\forall u, v$, $Pr[(u,v)$ *is cut* $] \le \frac{c \cdot d_G(u,v)}{\delta}\log n$.

Proof of Theorem 5. We fix vertex u, and assume w.l.o.g. $\varrho < \frac{\delta}{4}$ ($\frac{c\varrho}{\delta} > 1$ if $\varrho \ge \frac{\delta}{4}$). Let $v_1, v_2, \ldots, v_n$ be an ordering of V s.t. $d(v_1,u) \le d(v_2,u), \le \ldots \le d(v_n,u)$. Let A_i be the event that v_i "first" cuts $B(u,\varrho)$, i.e.,

1. $B(v_i,\theta) \cap B(u,\varrho) \ne \emptyset$
2. $B(v_i,\theta) \cap B(u,\varrho) \ne B(u,\varrho)$
3. $\forall j < \arg\sigma(i), v_{\sigma(j)}$ does not cur or captures $B(u,\varrho)$.

As shown in Figure 3, obviously A_i is possible only when $\theta = a \in [d(v_i,u)-\varrho, d(v_i,u)+\varrho]$, and note that σ is a random permutation, so

$$Pr[A_i|\theta = a] \le \frac{1}{i}, \forall a \in [d(v_i,u)-\varrho, d(v_i,u)+\varrho]$$

recall that $\theta \in [\frac{\delta}{4}, \frac{\delta}{2}]$, thus

$$Pr[A_i] \le \frac{1}{i}\cdot\frac{2\varrho}{\delta/4} \le \frac{8}{i}\cdot\frac{\varrho}{\delta}$$

Therefore, $Pr[B(u,\varrho)$ is cut $] \le \sum_{i=1}^{n} Pr[A_i] \le \frac{8\varrho}{\delta}\log n$. □

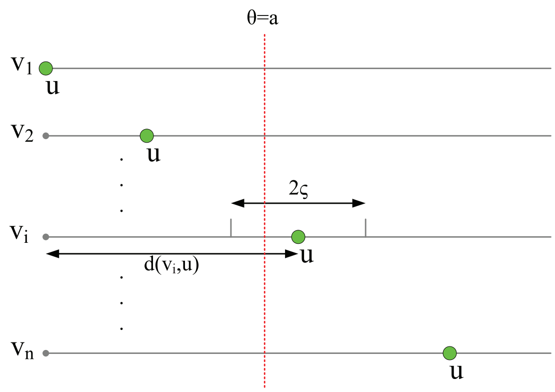

Figure 3: Proof of Theorem 5.

Actually we are able to provide an even tighter bound for $Pr[B(u,\varrho)$ is cut $]$. By observation, v_i may cut $B(u,\varrho)$ only when $d(v_i,u) \in [\frac{\delta}{4}-\varrho, \frac{\delta}{2}+\varrho]$, therefore,

$$\begin{aligned} Pr[B(u,\varrho) \textit{ is cut}] &\leq \sum_{i:d(v_i,u)\in[\delta/4-\varrho,\delta/2+\varrho]} \frac{1}{i}\cdot\frac{8\varrho}{\delta} \\ &= \frac{8\varrho}{\delta}\cdot\log\frac{|B(u,\delta/2+\varrho)|}{|B(u,\delta/4-\varrho)|} \\ &\leq \frac{8\varrho}{\delta}\cdot\log\frac{|B(u,\delta)|}{|B(u,\delta/8)|} \end{aligned}$$

2.2 Incorporate CKR decomposition into Tree Embedding Algorithm

By applying CKR Graph Decomposition procedure to decompose G into $G_1, G_2, \ldots, G_h$, at the second step in the Tree Embedding Algorithm, we are able to improve the bound $E(d_T(u,v)) = O(\log\Delta(G))d_G(u,v)$ in Lemma 4 to $E(d_T(u,v)) = O(\log n)d_G(u,v)$:

$$\begin{aligned} E[d_T(u,v)] &\leq 4\cdot\Delta(G)\cdot Pr[A_1] + 4\cdot\frac{\Delta(G)}{2}\cdot Pr[A_2|\bar{A}_1] \\ &\quad + \ldots + 4\cdot\frac{\Delta(G)}{2^i}\cdot Pr[A_i|\cap_{j=1}^{i-1}\bar{A}_j] + \ldots \\ &\leq 4\cdot\Delta(G)\cdot\frac{8d_G(u,v)}{\Delta(G)/2}\cdot\log\frac{|B(u,\Delta(G)/2)|}{|B(u,\Delta(G)/16)|} \\ &\quad +4\cdot\frac{\Delta(G)}{2}\cdot\frac{8d_G(u,v)}{\Delta(G)/4}\cdot\log\frac{|B(u,\Delta(G)/4)|}{|B(u,\Delta(G)/32)|} \\ &\quad +\ldots \\ &\quad +4\cdot\frac{\Delta(G)}{2^i}\cdot\frac{8d_G(u,v)}{\Delta(G)/2^{i+1}}\cdot\log\frac{|B(u,\Delta(G)/2^{i+1})|}{|B(u,\Delta(G)/2^{i+4})|} \\ &\quad +\ldots \\ &\leq 64\cdot d_G(u,v)\cdot 3\log n \\ &= O(\log n)\cdot d_G(u,v) \end{aligned}$$

Up to now, we have shown the correctness of the sc Tree Embedding Algorithm, thus Theorem 1 has been finally proved.

References

[1] Yair Bartal. Probabilistic approximations of metric spaces and its algorithmic applications. *IEEE Symposium on Foundations of Computer Science,* pages 184-193, 1996.

[2] Jittat Fakcharoenphol, Satish Rao and Kunal Talwar. Approximating metrics by tree metrics. *ACM SIGACT News,* Volume 35, Issue 2, pages 60-70, June 2004.