1 l_1 Embeddings and Sparsest Cut

Metric embeddings are a powerful tool in a variety of settings and they got their impetus in computer science with the application to sparsest cut. In this section, we focus on l_1-embeddings of finite metrics for their application to sparsest cut. That is, we wish to embed a finite metric (V, d) into R^h for some h to minimize distortion.

Suppose every n point metric is embeddable into l_1 with distortion $\alpha(n)$. We will present an algorithm which is an $\alpha(n)$-approximation of the solution to the LP relaxation. This is based on the characterization of l_1 metrics as those expressible as positive sum of cut-metrics.

Theorem 1 *Let (x^*, y^*) be an optimal solution to LP (OPT_{LP}) for sparsest cut on $G = (V, E)$ and $s_1t_1, s_2t_2, \ldots, s_kt_k$ with demand $dem(i)$ for $1 \leq i \leq k$. Let d be the metric induced by x^* on V. If d can be embedded into l_1 with distortion $\alpha(n)$, then there is a sparsest cut S^* such that*

$$\frac{c(S^*)}{dem(S^*)} \leq \alpha(n)\mathrm{OPT}_{LP}. \tag{1}$$

Moreover, if the l_1-embedding f can be computed in polynomial-time then S^* can be found in polynomial-time.

Proof: Let f be an l_1-embedding of (V, d) with distortion $\alpha(n)$. From the definition, OPT_{LP} can be represented as

$$\mathrm{OPT}_{LP} = \frac{\sum_{(u,v)\in E} c(u,v)d(u,v)}{\sum_{i=1}^{k} dem(i)d(s_i,t_i)} \tag{2}$$

Let $d'(u,v) = ||f(u) - f(v)||_1$. Without loss of generality, assume that f is an expansion, such that $d(u,v) \leq d'(u,v) \leq \alpha(n)d(u,v)$.

$$(2) \geq \frac{1}{\alpha(n)} \frac{\sum_{(u,v)\in E} c(u,v)d'(u,v)}{\sum_{i=1}^{k} dem(i)d'(s_i,t_i)} \tag{3}$$

Since it is an l_1-embedding, there is a cut-cone represention. Let $\lambda : 2^V \rightarrow R^+$ be the cut cone representation of d'. Then,

$$\begin{aligned}
(3) &= \frac{1}{\alpha(n)} \frac{\sum_{(u,v)\in E} c(u,v) \sum_S \lambda_S d_S(u,v)}{\sum_{i=1}^{k} dem(i) \sum_S \lambda_S d_S(s_i,t_i)} && (4)\\
&= \frac{1}{\alpha(n)} \frac{\sum_S \lambda_S \sum_{(u,v)\in\delta(S)} c(u,v)}{\sum_S \lambda_S \sum_{(s_i,t_i)\in\delta(S)} dem(i)} && (5)\\
&\geq \frac{1}{\alpha(n)} \min_S \frac{c(S)}{dem(S)} && (6)\\
&= \frac{1}{\alpha(n)} \min_{S:\lambda_S>0} \frac{c(S)}{dem(S)} && (7)
\end{aligned}$$

Therefore, $\text{OPT}_{LP} \geq \frac{1}{\alpha(n)} \min_S \frac{c(S)}{dem(S)}$ which implies $\frac{c(S^*)}{dem(S^*)} \leq \alpha(n)\text{OPT}_{LP}$.

□

Note however that it does not immediately give a polynomial time algorithm. We will introduce the specific embeddings of Bourgain to derive a randomized polynomial time algorithm. For detailed proof of this theorem, refer to [1].

Theorem 2 (Bourgain) *Given a metric space (V, d), where $|V| = n$, there is a ramdomized polynomial-time algorithm to compute $f : V \to R^{O(\log^2 n)}$ such that $\forall u, v \in V$*

$$d(u, v) \geq ||f(u) - f(v)||_1 \geq \frac{d(u, v)}{C \log n} \quad (8)$$

for some constant C.

The theorem produces an l_1-embedding with $O(\log n)$ distortion in polynomial-time.

Therefore, the polynomial-time algorithm for finding sparsest cut would be

1. Solve LP to get distance $d : V \times V \to R^+$.
2. Obtain l_1-embedding using the Bourgain's theorem.
3. For each dimension, convert the l_1-embedding to cut-metric representation.
4. Pick the best cut.

Lastly, we complete this section by stating Rao's result on planar graphs [2].

Theorem 3 (Rao 99') *Every planar graph metric on n vertices can be embedded into l_1 with distortion $O(\sqrt{\log n})$.*

2 Tree Metrics

Tree metrics are convenient tools for analyzing and providing solutions to graph related problems. By converting a graph to a tree, we can reduce measurment costs. We have seen such examples in Steiner tree and Gomory-Hu tree.

Definition A metric (V, d) is a tree metric if there exists a tree $T = (V, E_T)$ with non-negative edge length such that $d(u, v) = d_T(u, v)$.

Here $d_T(u, v)$ is the distance in the tree between u and v, defined to be the sum of the length of the edges on the unique (u, v)-path in tree T.

Question Given a metric (V, d) induced by an edge-weighted graph $G = (V, E)$, is there a tree metric (V, d') that approximates (V, d)?

Claim 4 $E[d_T(u, v)] \leq 2d_G(u, v)$, $\forall u, v$

Theorem 5 *Given an undirected edge-weighted graph $G = (V, E)$, there is a randomized polynomial-time algorithm that produces a random edge-weighted tree $T = (V_T, E_T)$ such that*

1. $V \subseteq V_T$
2. $d_G(u,v) \leq d_T(u,v)$, $\forall u, v$
3. $E[d_T(u,v)] \leq O(\log |V|) d_G(u,v)$, $\forall u, v$

Theorem 6 *Given G, there exists a finite collection of tree metric $\mathcal{T} = \{T_1, \ldots, T_e\}$ and a distribution μ on them such that*

1. $\forall T \subseteq \mathcal{T}$, *$T$ dominates G*
2. $\forall u, v$, $\sum_{T \in \mathcal{T}} \mu(T) d_T(u,v) \leq O(log|V|) d_G(u,v)$

References

[1] J. Bourgain. On lipschitz embedding of finite metric spaces in hilbert space. *Israel Journal of Mathematics*, 52(1):46–52, March 1985.

[2] Tom Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *J. ACM*, 46(6):787–832, November 1999.