1 Network Design Problem

In the *Abstract Network Design Problem (ANDP)* a graph $G = (V, E)$, a nonnegative edge costs $c_e : E \to \mathbb{R}^+$ and a function $f : 2^V \to \{0, 1\}$ are given. The goal is to find a subset $E' \subseteq E$ such that $S \subseteq V$ implies $\delta_{E'}(S) \geq f(S)$. Here is ANDP modeled by the an integer program:

$$\min \sum_{e \in E} c_e x_e$$
$$\sum_{e \in \delta(S)} x_e \geq f(S), \forall S \subseteq V$$
$$x_e \in \{0, 1\}, \forall e \in E$$

In general function $f : 2^V \to \{0, 1\}$ can be any arbitrary function. However, interesting approximation algorithms can be found for this problem if f has certain properties. So, we spend some time on defining such properties. In all of the following definitions $f : 2^V \to \{0, 1\}$ is a function and $A, B \subset V$.

Definition 1 f is *maximal* if for all disjoint A and B we have $f(A \cup B) \leq \max\{f(A), f(B)\}$.

Definition 2 f is *proper* if it is symmetric, maximal and $f(V) = 0$.

Definition 3 f is *downward monotone* if $f(A) \leq f(B)$ for all $\emptyset \neq B \subset A$.

Definition 4 f is *skew-supermodular* if for all A and B one of the following conditions hold,

1. $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$
2. $f(A) + f(B) \leq f(A - B) + f(B - A)$

Claim 1 *If f is both proper and downward monotone, then it is also skew-supermodular.*

Proof: Since f is downward monotone, $A - B \subset A$ and $B - A \subset B$ we get:

$$f(A) \leq f(A - B)$$
$$f(B) \leq f(B - A)$$

so the second condition of skew-supermodularity always holds. □

Example 1 *The requirement function in the Steiner Forest problem is proper.*

Example 2 *Let $X = \{s_1, \ldots, s_k\}$ and $Y = \{t_1, \ldots, t_k\}$ be two disjoint sets of terminals for the point to point connection problem, then $f(A) = 1$ if and only if $|A \cap X| \neq |A \cap Y|$*

Example 3 *Lower Capacitated Tree Problem: Given $G = (V, E)$, $c : E \to \mathbb{R}^+$ and a $k \in \mathbb{Z}^+$ find a set $E' \subseteq E$ of minimum cost such that every connected component in $G[E']$ has at least k edges; $f(A) = 1 \Leftrightarrow |A| < k$.*

The above function is downward monotone but not proper.

Example 4 *Connectivity Augmentation: Given an undirected k-edge connected graph $G = V, E$ and a set of edges $E^{aug} \subseteq V \times V - E$, find a set $E' \subseteq E^{aug}$ of minimum cost such that $G = (V, E \cup E')$ is $(k+1)$-edge connected; $f(A) = 1 \leftrightarrow \delta_E(A) = k$.*

This function is proper but not skew-supermodular.

Definition 5 *Let $G = (V, E)$ and $f : 2^V \to \mathbb{Z}^+$. For each $X \subseteq E$, the residual requirement function $f_X : 2^V \to \mathbb{Z}^+$ is defined as:*

$$f_X(A) = \max\{0, f(A) - |\delta_X(A)|\}$$

Claim 2 *If f is skew-supermodular then f_X is also skew-supermodular for each $X \subseteq E$.*

Proof: Obviously, the lemma follows if the following inequalities hold ($|\delta_X(.)|$ is submodular).

$$|\delta_X(A)| + |\delta_X(B)| \geq |\delta_X(A - B)| + |\delta_X(B - A)|$$
$$|\delta_X(A)| + |\delta_X(B)| \geq |\delta_X(A \cap B)| + |\delta_X(B \cup A)|$$

In the first case, if an edge has one end in $A \cap B$ and the other end in $\overline{A \cup B}$ it only contributes to the left. Other edges equally contribute to both sides. Similarly, in the second case, if an edge has one end in $A - B$ and the other end in $B - A$, it only contributes to the left. Other edges equally contribute to both sides.

□

Claim 3 *If f is a $\{0, 1\}$ skew-supermodular function and $X \subseteq E$, then any minimal violated sets with respect to X are disjoint.*

Proof: Let Y and Z be two minimal violated sets; that is: $f(Y) = f(Z) = 1$ and $\delta_X(Y) = \delta_X(Z) = \emptyset$. From the last equality we get:

$$\delta_X(Y - Z) = \delta_X(Z - Y) = \delta_X(Z \cap Y) = \delta_X(Y \cup Z) = \emptyset$$

On the other hand from the skew-supermodularity we know at least one of the following conditions holds:

$$f(Z - Y) = f(Y - Z) = 1$$
$$f(Z \cap Y) = f(Y \cup Z) = 1$$

In either case if Z and Y are not disjoint then the their minimality assumption will be violated.

□

Now, we analyze the ANDP for a $\{0, 1\}$ skew-supermodular function $f : 2^V \to \{0, 1\}$. We also assume that f is specified as an oracle, that for each $A \subseteq V$ the oracle return $f(A)$. The primal and dual relaxed LPs for the problem follow, respectively.

Primal:

$$\min \sum_{e\in E} c_e x_e$$
$$\sum_{e\in\delta(S)} x_e \geq f(S), \forall S \subseteq V$$
$$x_e \geq 0, \forall e \in E$$

Dual:

$$\max \sum_{S\subset V} f(S) y_S$$
$$\sum_{S:e\in\delta(S)} y_S \leq c_e, \forall e \in E$$
$$y_S \geq 0, \forall S \subseteq V$$

Algorithm 1 Primal-Dual Algorithm

$F \leftarrow \emptyset$
while f is not feasible **do**
  Let $C_1, \ldots, C_k$ be the minimal violated sets w.r.t F {they are disjoint}
  for all $i \in \{1, \ldots, k\}$ **do**
    Raise y_{C_i} uniformly for all C_is until $\exists j, e \in \delta(C_j), \sum_{S:e\in\delta(C_j)} y_{C_j} = c_e$
  end for
  $F \leftarrow F + e$
end while
{Do reverse delete}
Let $e_1, \ldots, e_\ell$ be added in order
for $h \leftarrow \ell$ downto 1 **do**
  if $F - e_h$ is feasible **then**
    $F \leftarrow F - e_h$
  end if
end for

Theorem 4 *The Primal-Dual algorithm gives a 2-approximation for any $\{0,1\}$ skew-supermodular function f.*

Proof: Let F' be the output of the algorithm. We like to show $c(F') \leq 2.\sum_{S\in\mathcal{S}} f(S)y_S$. Since every picked edge is tight, we know $c(F') = \sum_{e\in F'} c_e = \sum_{e\in F'}\sum_{S:e\in\delta(S)} y_S$. Moreover, $\sum_{e\in F'}\sum_{S:e\in\delta(S)} y_S = \sum_{S\in\mathcal{S}} y_S \delta_{F'}(S)$, where $\delta_{F'}(S)$ denotes the number of edges of F' cross the $(S, V\backslash S)$ cut.

Let A_i contains all minimally violated sets at the ith iteration. Also, let Δ_i be the dual growth in this iteration. We rewrite the above sum as $\sum_{S\in\mathcal{S}} y_S\delta_{F'}(S) = \sum_{i=1}^{\alpha}\sum_{S\in A_i} \Delta_i deg_{F'}(S)$, where α is the number of iterations. Using the result of the next lemma, that is $\sum_{S\in A_i} deg_{F'}(S) \leq 2|A_i|$, we have:

$$c(F') = \sum_{e \in F'} c_e = \sum_{e \in F'} \sum_{S: e \in \delta(S)} y_S = \sum_{S \in \mathcal{S}} y_S \delta_{F'}(S) = \sum_{i=1}^{\alpha} \sum_{S \in A_i} \Delta_i deg_{F'}(S) \leq \sum_{i=1}^{\alpha} \Delta_i.2|A_i| \leq 2. \sum_{S \in \mathcal{S}} f(S) y_S$$

□

Lemma 5 *For any iteration i of the algorithm we have $\sum_{S \in A_i} deg_{F'}(S) \leq 2|A_i|$.*

Proof: Consider the graph $G[F']$, and fix some iteration i. In this graph contract each set of active vertices in iteration i to a single vertex (call it an active vertex). Similarly, contract inactive sets to inactive vertices. Let the resulting graph be H. We show that no vertex in H has degree less than two.

Suppose there exist a component C that has degree one in H. So there exists an edge crossing $(C, V \backslash C)$ cut. Since F' has no redundant edges (second phase of the algorithm), $f(C) \geq 1$, that is C is violating. □

2 Steiner Network Problem

Steiner Network Problem (also called Survivable Network Design) is defined as follows. Given $G = \{V, E\}$, $c : E \rightarrow \mathbb{R}^+$ and a requirement $r_{uv} \in \mathbb{Z}^+ \cup \{0\}$ for each pair $(u, v) \in V \times V$, find a set $E' \subseteq E$ of minimum cost such that $G[E']$ has r_{uv} disjoint paths between u and v.

This is special case of the network design problem where $f(A) = \max_{(u,v):|A \cap \{u,v\}|=1} r_{uv}$. Here is a simple $2\max(r_{uv})$-approximation which is result of augmenting Steiner Forest approximation algorithm.

Algorithm 2 Steiner Network $2\max(r_{uv})$-Approx

Find a 2-approx solution for the Steiner Forest Problem
for $i \leftarrow 2$ to $\max(r_{uv})$ **do**
  Augment the solution for those who want i-connectivity
end for

The approximation ratio of this algorithm can be reduced to $2\log(\max(r_{uv}))$ if we do the augmentation in the reverse order.