Spring 2009, CS 598CSC: Approximation Algorithms
Homework 5
Due: 04/22/2009 in class

Instructions and Policy: You are not allowed to consult any material outside of the textbook and class notes in solving these problems. Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince us that you know the solution, as quickly as possible.

Problem 1 [15 pts] In the k-tree problem you are given an undirected edge-weighted graph $G = (V, E)$ with edge weights $c : E \to \mathcal{R}^+$ and an integer k. The goal is to find a tree $T = (V_T, E_T)$ in G of smallest edge weight ($\sum_{e \in E_T} c(e)$) such that $|V_T| \geq k$. Show that if there is an α-approximation for k-tree then there is an α-approximation for the Steiner tree problem. Recall that in the Steiner tree problem, the input is an edge-weighted graph $G = (V, E)$ and a set of terminals $S \subseteq V$; the goal is to find a tree T of minimum edge-weight that connects (contains) all the terminals S.

Problem 2 [55 pts] In this problem you will derive an $O(\log k \cdot \log n)$ approximation for the rooted k-Steiner-tree problem which is related to the previous problem. The input consists of an edge-weighted undirected graph $G = (V, E)$, a specified root vertex r and a set $S \subset V$ of terminals. The goal is to find a min-cost tree (V_T, E_T), a sub-graph of G, such that $r \in V_T$ and $|S \cap V_T| \geq k$. Obtain an approximation for this problem following the outline below.

- Consider the density variant of the problem where the goal is to find a tree $T = (V_T, E_T)$ rooted at r that minimizes the ratio $c(E_T)/|V_T \cap S|$. Write an LP relaxation for this problem using ideas similar to the one for Sparsest Cut (and Steiner tree). Using the scaling ideas that we used to reduce Sparsest Cut to Multicut, reduce the density variant of k-Steiner-tree to the Steiner tree problem and obtain an $O(\log n)$ approximation. Recall that the Steiner tree LP has an integrality gap of 2.
- Use an approximation algorithm for the density variant above in an iterative greedy fashion to create a tree rooted at r with at least k terminals. What is the density of this tree when compared to the density of the optimal solution to the original k-Steiner-tree problem?
- If the tree you have in the previous step has many more than k terminals, *prune* it to have k' terminals where $k \leq k' \leq 2k$ such that the density of the resulting tree is not much worse than the tree you started with.
- How can you connect the pruned tree to the original root r without incurring too much cost? Assuming you know the optimal cost, can you preprocess the instance to ensure that this connection cost is not too much?

Problem 3 [15 pts] Prove that any ring metric isometrically embeds into ℓ_1.

Problem 4 [15 pts] Given a graph $G = (V, E)$ with edge-weights $c : E \to \mathcal{R}^+$, you wish to partition G into $G_1 = G[V_1], G_2 = G[V_2], G_3 = G[V_3]$ such that $\lfloor |V|/3 \rfloor \leq |V_i| \leq \lceil |V|/3 \rceil$ for $1 \leq i \leq 3$, and the cost of the edges between the partitions is minimized. Using an α-approximation for the sparsest cut problem, give a pseudo-approximation for this problem where you partition the graph into 3 pieces $G[V_1'], G[V_2'], G[V_3']$ such that $|V|/c_2 \leq |V_i'| \leq |V|/c_1$ for some constants $1 < c_1 < c_2$ and the cost of the edges between the partitions is $O(\alpha)\text{OPT}$. What constants c_1, c_2 can you guarantee? Note that c_1 and c_2 should be *constants*, independent of the graph size. (Hint: this problem is similar to the one on partitioning into two pieces that is in the book on applications of sparsest cut (Section 21.6.3).)