Chapter 17

Random Walks IV

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> "Do not imagine, comrades, that leadership is a pleasure! On the contrary, it is a deep and heavy responsibility. No one believes more firmly than Comrade Napoleon that all animals are equal. He would be only too happy to let you make your decisions for yourselves. But sometimes you might make the wrong decisions, comrades, and then where should we be? Suppose you had decided to follow Snowball, with his moonshine of windmills-Snowball, who, as we now know, was no better than a criminal?"
>
> – Animal Farm, George Orwell.

17.1 Cover times

We remind the reader that the cover time of a graph is the expected time to visit all the vertices in the graph, starting from an arbitrary vertex (i.e., worst vertex). The cover time is denoted by $\mathcal{C}(\mathsf{G})$.

Theorem 17.1.1 *Let* G *be an undirected connected graph, then* $\mathcal{C}(\mathsf{G}) \leq 2m(n-1)$*, where* $n = |V(\mathsf{G})|$ *and* $m = |E(\mathsf{G})|$.

Proof: (Sketch.) Construct a spanning tree T of G, and consider the time to walk around T. The expected time to travel on this edge on both directions is $\mathbf{CT}_{uv} = \mathtt{h}_{uv} + \mathtt{h}_{vu}$, which is smaller than $2m$, by Lemma 17.5.1. Now, just connect up those bounds, to get the expected time to travel around the spanning tree. Note, that the bound is independent of the starting vertex. ∎

Definition 17.1.2 The ***resistance*** of G is $\mathbf{R}(\mathsf{G}) = \max_{u,v \in V(\mathsf{G})} \mathbf{R}_{uv}$; namely, it is the maximum effective resistance in G.

Theorem 17.1.3 $m\mathbf{R}(\mathsf{G}) \leq \mathcal{C}(\mathsf{G}) \leq 2e^3 m\mathbf{R}(\mathsf{G}) \ln n + 2n$.

Proof: Consider the vertices u and v realizing $\mathbf{R}(\mathsf{G})$, and observe that $\max(\mathrm{h}_{uv}, \mathrm{h}_{vu}) \geq \mathbf{CT}_{uv}/2$, and $\mathbf{CT}_{uv} = 2m\mathbf{R}_{uv}$ by Theorem 17.5.2. Thus, $\mathcal{C}(\mathsf{G}) \geq \mathbf{CT}_{uv}/2 \geq m\mathbf{R}(\mathsf{G})$.

As for the upper bound. Consider a random walk, and divide it into ***epochs***, where a epoch is a random walk of length $2e^3 m\mathbf{R}(\mathsf{G})$. For any vertex v, the expected time to hit u is $\mathrm{h}_{vu} \leq 2m\mathbf{R}(\mathsf{G})$, by Theorem 17.5.2. Thus, the probability that u is not visited in a epoch is $1/e^3$ by the Markov inequality. Consider a random walk with $\ln n$ epochs. We have that the probability of not visiting u is $\leq (1/e^3)^{\ln n} \leq 1/n^3$. Thus, all vertices are visited after $\ln n$ epochs, with probability $\geq 1 - 1/n^3$. Otherwise, after this walk, we perform a random walk till we visit all vertices. The length of this (fix-up) random walk is $\leq 2n^3$, by Theorem 17.1.1. Thus, expected length of the walk is $\leq 2e^3 m\mathbf{R}(\mathsf{G}) \ln n + 2n^3(1/n^2)$. ∎

17.1.1 Rayleigh's Short-cut Principle.

Observe that effective resistance is never raised by lowering the resistance on an edge, and it is never lowered by raising the resistance on an edge. Similarly, resistance is never lowered by removing a vertex.

Another interesting fact, is that effective resistance comply with the triangle inequality.

Observation 17.1.4 *For a graph with minimum degree d, we have $\mathbf{R}(\mathsf{G}) \geq 1/d$ (collapse all vertices except the minimum-degree vertex into a single vertex).*

Lemma 17.1.5 *Suppose that G contains p edge-disjoint paths of length at most ℓ from s to t. Then $\mathbf{R}_{st} \leq \ell/p$.*

17.2 Graph Connectivity

Definition 17.2.1 A ***probabilistic log-space Turing machine*** for a language L is a Turing machine using space $O(\log n)$ and running in time $O(\mathrm{poly}(n))$, where n is the input size. A problem A is in RLP, if there exists a probabilistic log-space Turing machine M such that M accepts $x \in L(A)$ with probability larger than $1/2$, and if $x \notin L(A)$ then $M(x)$ always reject.

Theorem 17.2.2 *Let* USTCON *denote the problem of deciding if a vertex s is connected to a vertex t in an undirected graph. Then $USTCON \in$* RLP.

Proof: Perform a random walk of length $2n^3$ in the input graph G, starting from s. Stop as soon as the random walk hit t. If u and v are in the same connected component, then $\mathrm{h}_{st} \leq n^3$. Thus, by the Markov inequality, the algorithm works. It is easy to verify that it can be implemented in $O(\log n)$ space. ∎

Definition 17.2.3 A graph is ***d-regular***, if all its vertices are of degree d.

A d-regular graph is ***labeled*** if at each vertex of the graph, each of the d edges incident on that vertex has a unique label in $\{1, \ldots, d\}$.

Any sequence of symbols $\sigma = (\sigma_1, \sigma_2, \ldots)$ from $\{1, \ldots, d\}$ together with a starting vertex s in a labeled graph describes a ***walk*** in the graph. For our purposes, such a walk would almost always be finite.

A sequence σ is said to ***traverse*** a labeled graph if the walk visits every vertex of G regardless of the starting vertex. A sequence σ is said to be a ***universal traversal sequence*** of a labeled graph if it traverses all the graphs in this class.

Given such a universal traversal sequence, we can construct (a non-uniform) Turing machine that can solve USTCON for such d-regular graphs, by encoding the sequence in the machine.

Let $\mathcal{F}$ denote a family of graphs, and let $U(\mathcal{F})$ denote the length of the shortest universal traversal sequence for all the labeled graphs in $\mathcal{F}$. Let $\mathbf{R}(\mathcal{F})$ denote the maximum resistance of graphs in this family.

Theorem 17.2.4 $U(\mathcal{F}) \leq 5m\mathbf{R}(\mathcal{F}) \lg(n\,|\mathcal{F}|)$.

Let $U(d, n)$ denote the length of the shortest universal traversal sequence of connected, labeled n-vertex, d-regular graphs.

Lemma 17.2.5 *The number of labeled n-vertex graphs that are d-regular is $(nd)^{O(nd)}$.*

Proof: There are at most n^{nd} choices for edges in the graph. Every vertex has $d!$ possible labeling of the edges adjacent to it. ∎

Lemma 17.2.6 $U(d, n) = O\left(n^3 d \log n\right)$.

Proof: The diameter of every connected n-vertex, d-regular graph is $O(n/d)$. And so, this also bounds the resistance of such a graph. The number of edges is $m = nd/2$. Now, combine Lemma 17.2.5 and Theorem 17.2.4. ∎

This is, as mentioned before, not uniform solution. There is by now a known log-space deterministic algorithm for this problem, which is uniform.

17.2.1 Directed graphs

Theorem 17.2.7 *One can solve the $\overrightarrow{\text{STCON}}$ problem with a log-space randomized algorithm, that always output NO if there is no path from s to t, and output YES with probability at least $1/2$ if there is a path from s to t.*

17.3 Graphs and Eigenvalues

Consider an undirected graph $\mathsf{G} = \mathsf{G}(V, E)$ with n vertices. The adjacency matrix $\mathsf{M}(\mathsf{G})$ of G is the $n \times n$ symmetric matrix where $\mathsf{M}_{ij} = \mathsf{M}_{ji}$ is the number of edges between the vertices v_i and v_j. If G is bipartite, we assume that V is made out of two independent sets X and Y. In this case the matrix $\mathsf{M}(\mathsf{G})$ can be written in block form.

Since $\mathsf{M}(\mathsf{G})$ is symmetric, all its eigenvalues exists $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$, and their corresponding orthonormal basis vectors are $e_1, \ldots, e_n$. We will need the following theorem.

Theorem 17.3.1 (Fundamental theorem of algebraic graph theory.) *Let $\mathsf{G} = \mathsf{G}(V, E)$ be an n-vertex, undirected (multi)graph with maximum degree d. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathsf{M}(\mathsf{G})$ and the corresponding orthonormal eigenvectors are $e_1, \ldots, e_n$. The following holds.*

(i) *If* G *is connected then* $\lambda_2 < \lambda_1$.

(ii) *For* $i = 1, \ldots, n$, *we have* $|\lambda_i| \leq d$.

(iii) d *is an eigenvalue if and only if* G *is regular.*

(iv) *If* G *is* d*-regular then the eigenvalue* $\lambda_1 = d$ *has the eigenvector* $e_1 = \frac{1}{\sqrt{n}}(1, 1, 1, \ldots, 1)$.

(v) *The graph* G *is bipartite if and only if for every eigenvalue* λ *there is an eigenvalue* $-\lambda$ *of the same multiplicity.*

(vi) *Suppose that* G *is connected. Then* G *is bipartite if and only if* $-\lambda_1$ *is an eigenvalue.*

(vii) *If* G *is* d*-regular and bipartite, then* $\lambda_n = d$ *and* $e_n = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1, -1, \ldots, -1)$*, where there are equal numbers of* 1*s and* -1*s in* e_n.

17.4 Bibliographical Notes

A nice survey of algebraic graph theory appears in [Wes01] and in [Bol98].

17.5 Tools from previous lecture

Lemma 17.5.1 *For any edge* $(u \rightarrow v) \in E$, $\mathtt{h}_{uv} + \mathtt{h}_{vu} \leq 2m$.

Theorem 17.5.2 *For any two vertices* u *and* v *in* G*, the commute time* $\mathbf{CT}_{uv} = 2m\mathbf{R}_{uv}$.

Bibliography

[Bol98] B. Bollobas. *Modern Graph Theory*. Springer-Verlag, 1998.

[Wes01] D. B. West. *Intorudction to Graph Theory*. Prentice Hall, 2ed edition, 2001.