CS 598: Spectral Graph Theory. Lecture 7

The Unique Games Conjecture and SDP Duality.

Alexandra Kolla

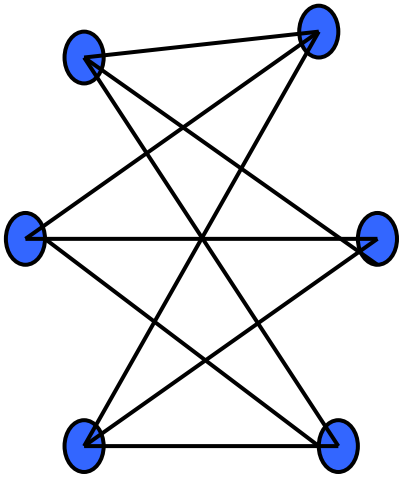
Today

- The Unique Games Conjecture
- Unique Games and Graphs
- SDP for Unique Games
- Duality proof that Random Unique Games are easy.

The MAX CUT Problem

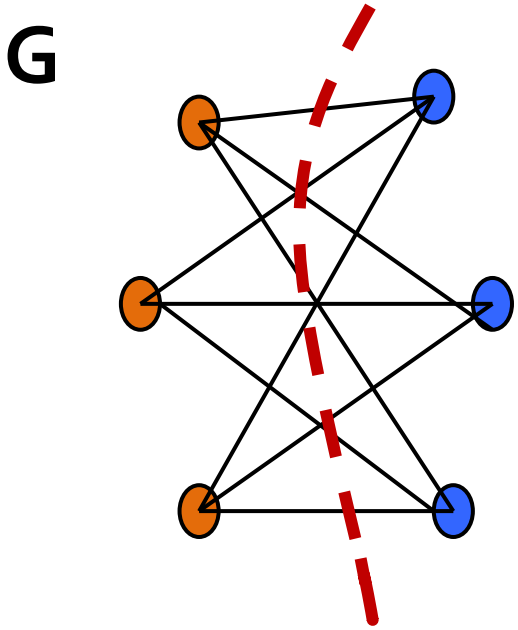
- **Input:** G = (V,E)

G



The MAX CUT Problem

- **Input:** G = (V,E)
- **Objective :** Partition G in (S,S') as to MAXIMIZE number of edges cut



- **[Karp '72]:** MAX CUT is NP-complete
- What about approximating MAX CUT?

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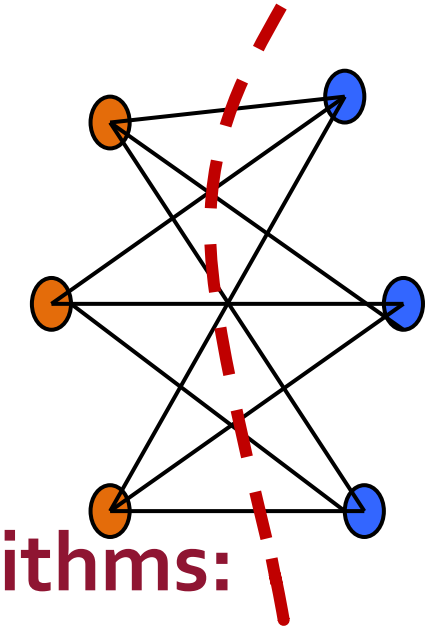

Approximation algorithms:

- **Random cut (trivial):** half of optimal
- **[GW'94]:** α_{GW}=0.878 approximation algorithm of MA

How many of you bet this is best we can do?

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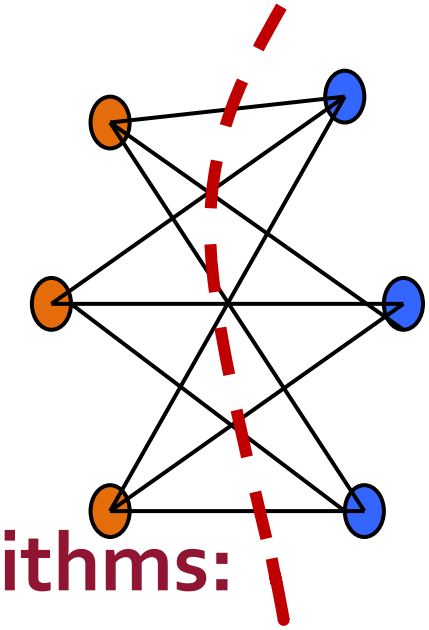

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If Unique Games Conjecture true, then it is!

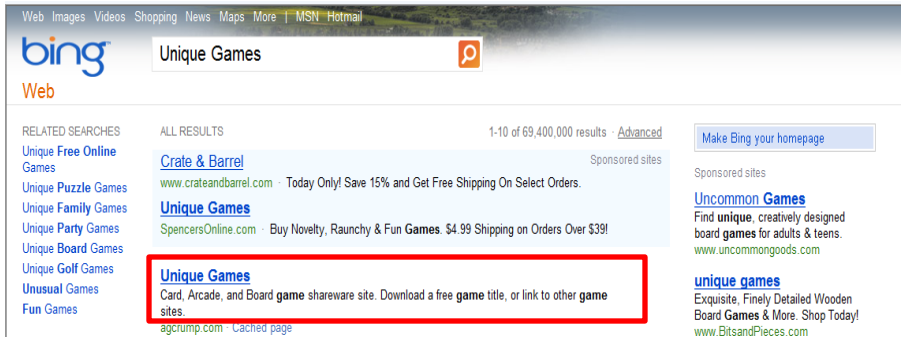
Can We Hope for Better Approximation Algorithms in P?

Previous inapproximability not a coincidence!
Unique Games Conjecture (UGC) captures
exact inapproximability of many more problems

Problem	Best Approximation Algorithm Known	UGC-Hardness
MaxCut	0.878[GW94]	0.878 [KKMO07]
Vertex Cover	2	2-ε [KR06]
Max k-CSP	$\Omega(k/2^k)$[CMM07]	$O(k/2^k)$[ST,AM,GR]

What are Unique Games?

1. Unique Games are popular not only among computer scientist!



bing: 70 million pages

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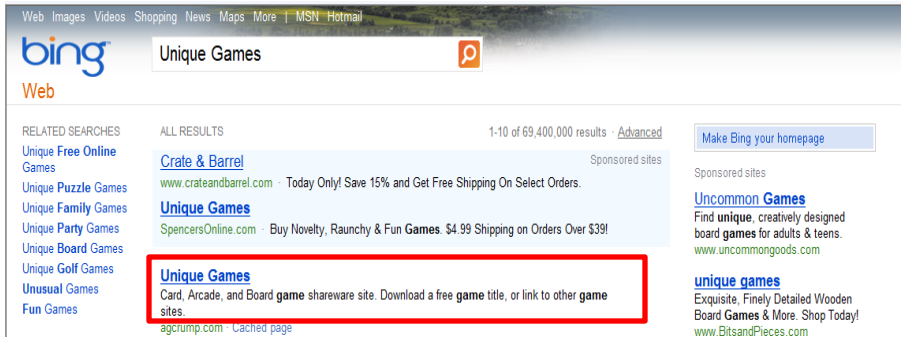

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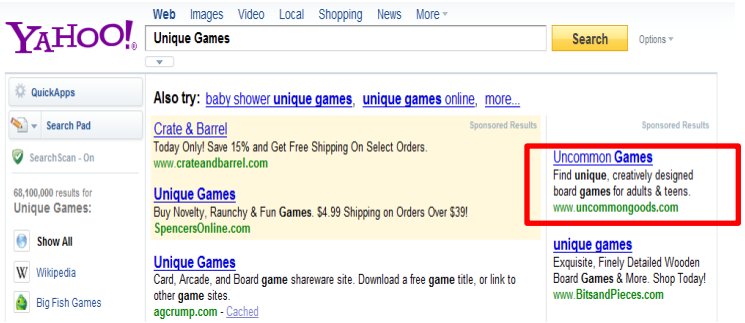

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2. We can purchase Unique Games on-line!

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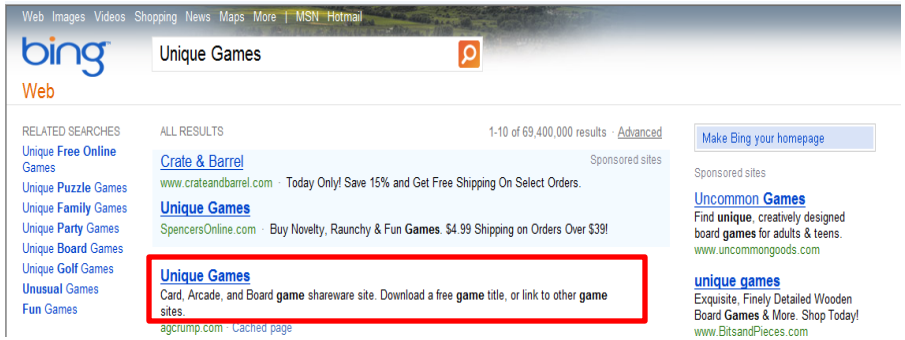

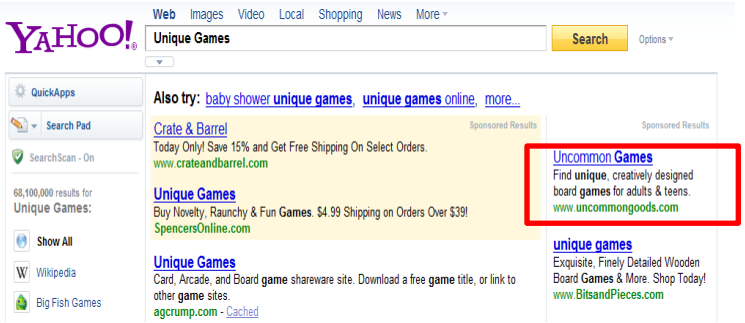

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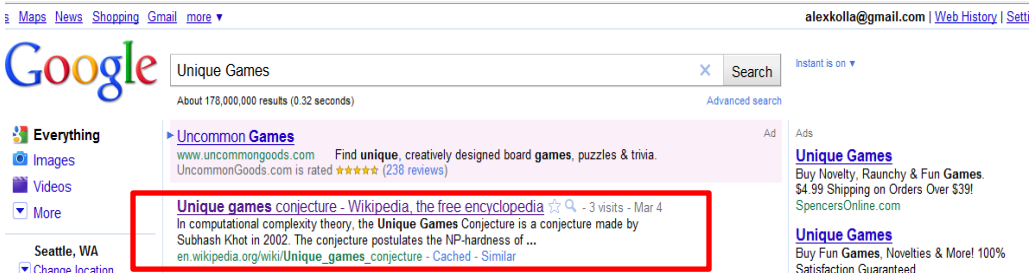
3. Unique Games are related to the Unique Games Conjecture…



Google: 178 million pages

Unique Games = Unique Label Cover Problem

Given: set of constraints

Linear Equations mod k :

$x_i - x_j = c_{ij} \bmod k$

GOAL

k ="alphabet" size

Find labeling that satisfies **maximum number of constraints.**

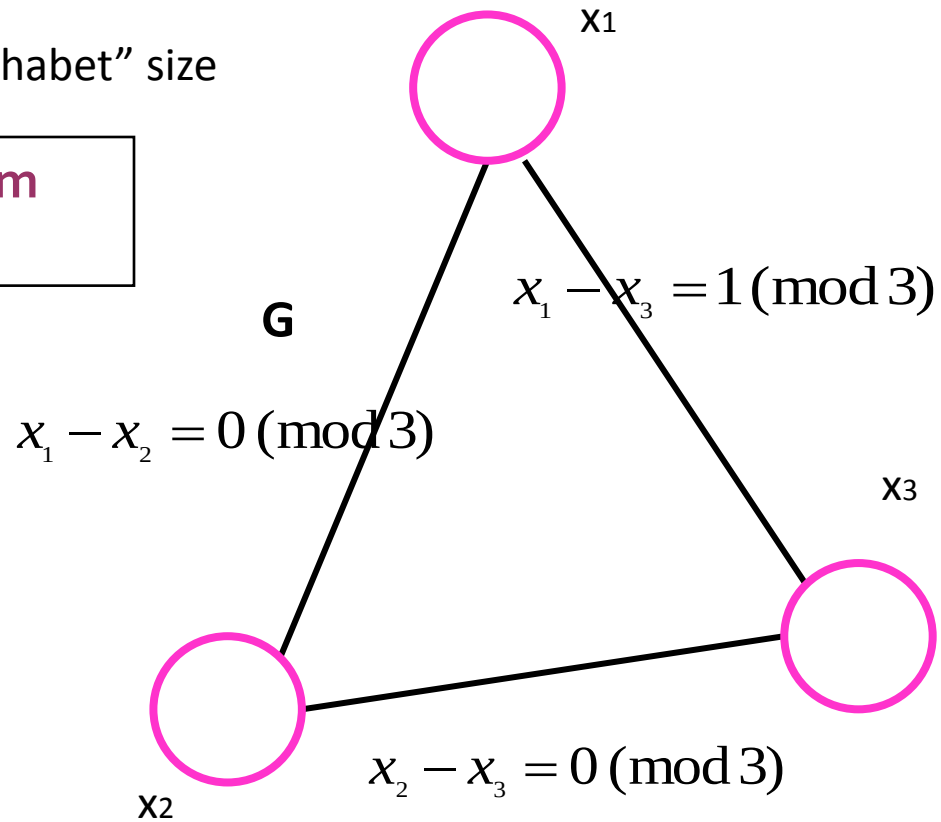
EXAMPLE

$$x_1 - x_2 = 0 \pmod 3$$

$$x_2 - x_3 = 0 \pmod 3$$

$$x_1 - x_3 = 1 \pmod 3$$

The constraint graph



Unique Games , an Example

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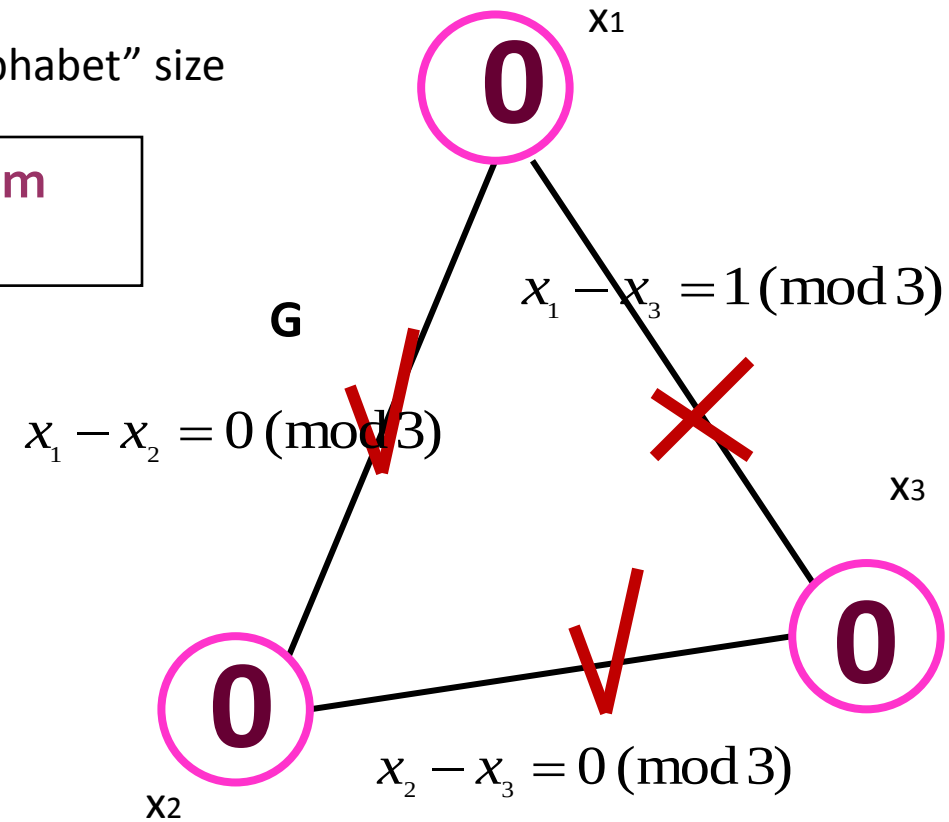

Satisfy 2/3 constraints

Unique Games , an Example

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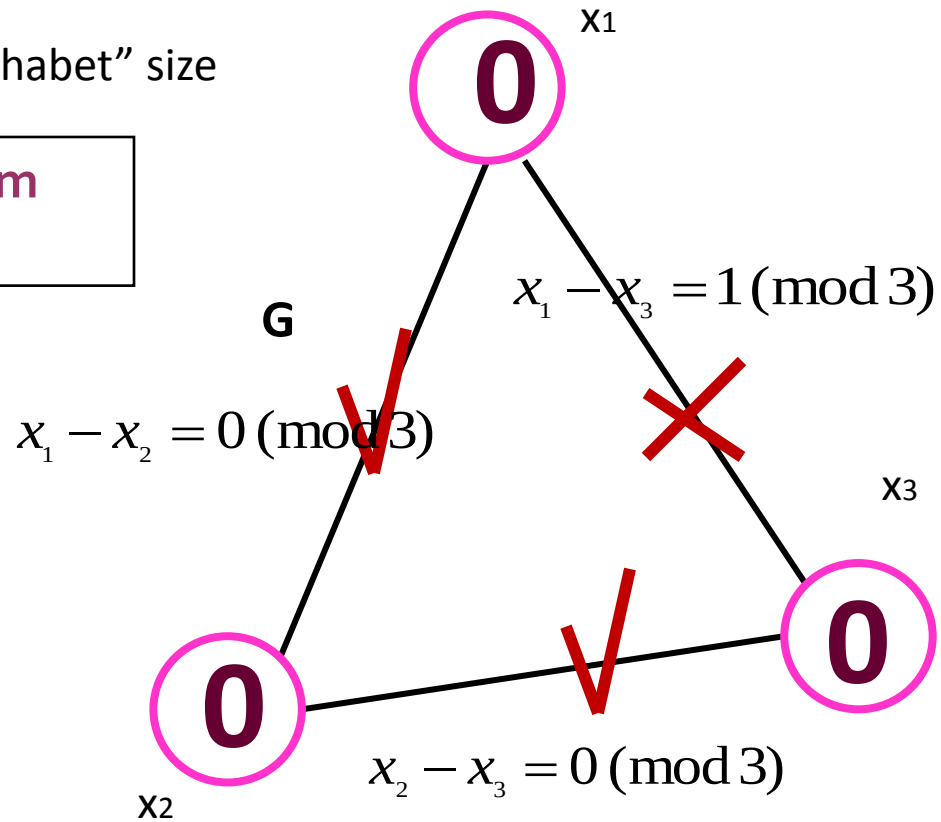

Rest of the talk: d-regular graphs

Unique Games Conjecture

- **[Khot'02]** For every positive ε and δ there is a large enough k s.t. for some instance of Unique Games with alphabet size k and OPT > $1 - \varepsilon$, it is NP hard to satisfy a δ fraction of all constraints.
- Implies: many known algorithms are optimal: MAX CUT, Vertex Cover, k-CSP,…
- **[Raghavendra'08]** SDP is the best method for any CSP problem assuming UGC!

Is Unique Games Conjecture True?

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Really embarassing to not know, since we can solve systems of linear equations !

Summary: Algorithmic Results for UG

	Algorithm	On 1-ε instances
General Graphs	Khot	$1-O(k^2 \varepsilon^{1/5}\sqrt{\log(1/\varepsilon)})$
	Trevisan	$1-O(\sqrt[3]{(\varepsilon \log n)})$
	Gupta-Talwar	$1-O(\varepsilon \log n)$
	CMM1	$k^{-\varepsilon/2-\varepsilon}$
	CMM2	$1-O(\varepsilon \sqrt{\log n}\sqrt{\log k})$
Special Graphs		
Expander	AKKTSV'08 KT'08,MM'10	Constant, depend on conductance
Local expander	AIMS'09, SR'09	Constant, depends on local expansion

SDP/LP based

Tight for SDP, there is counterexample

Almost all above approaches were LP or SDP based

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Few large eigenvalues	K'10	Quality ~~and running time~~ depends on eigenspace

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Purely SPECTRAL Approach "beats" SDP

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Key ingredient

ABS'10: Subexponential time algorithm for ANY instance

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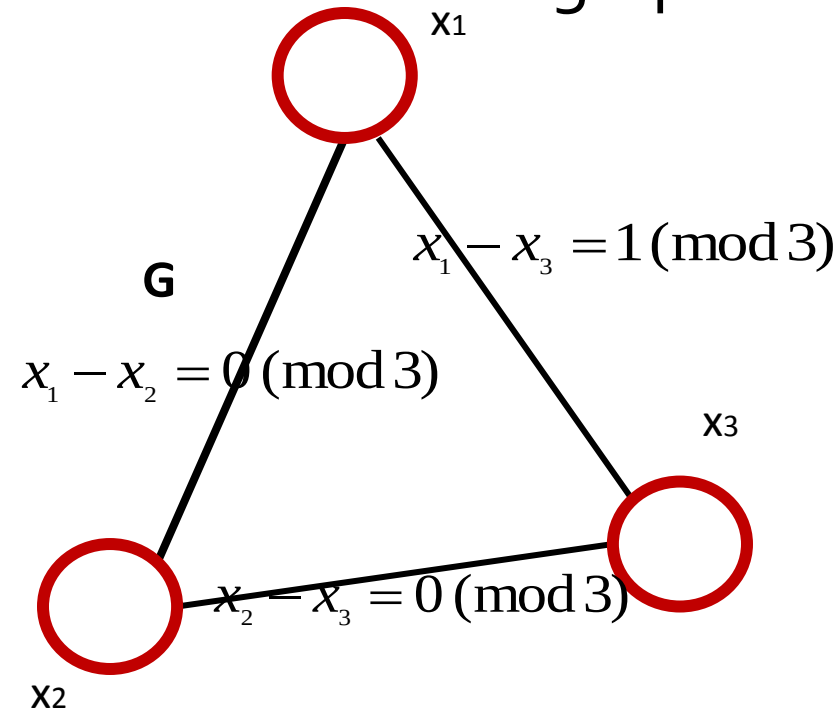
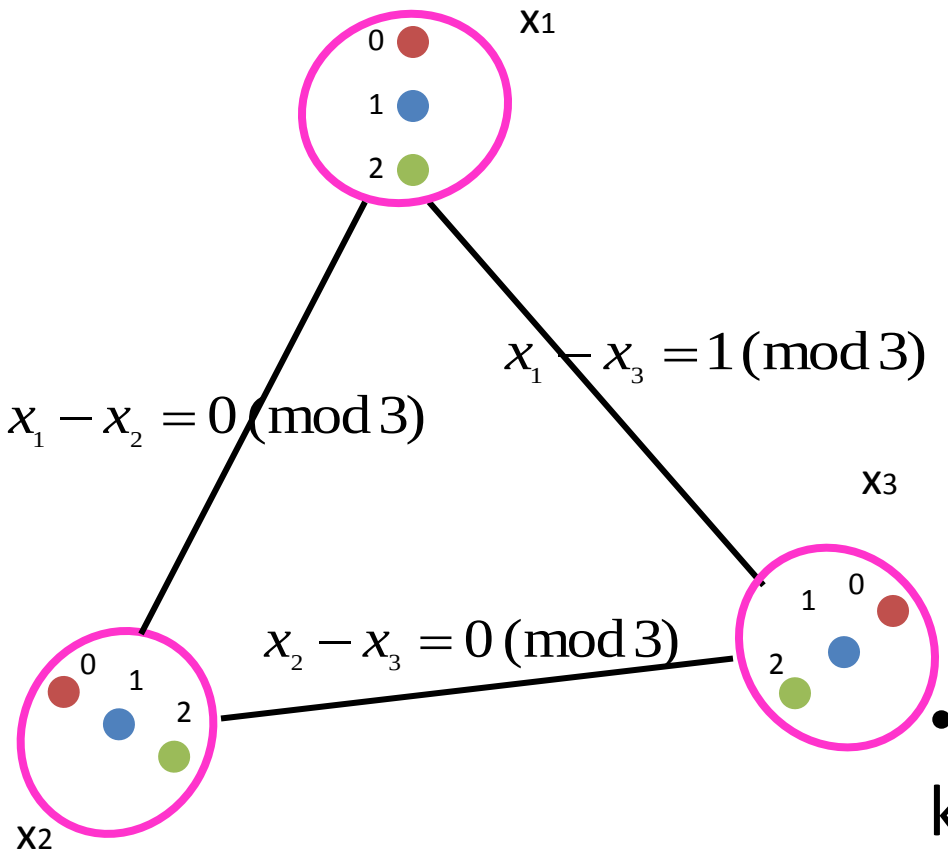
KMM'10: Semi-Random instances are easy

Unique Games and Graphs

1. The "constraint graph"



2. The "label-extended" graph



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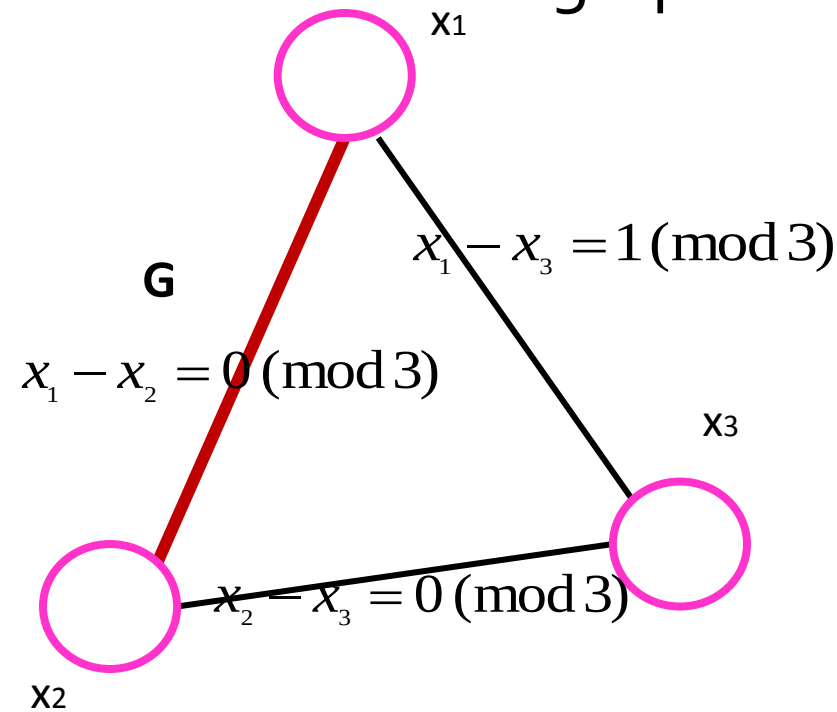

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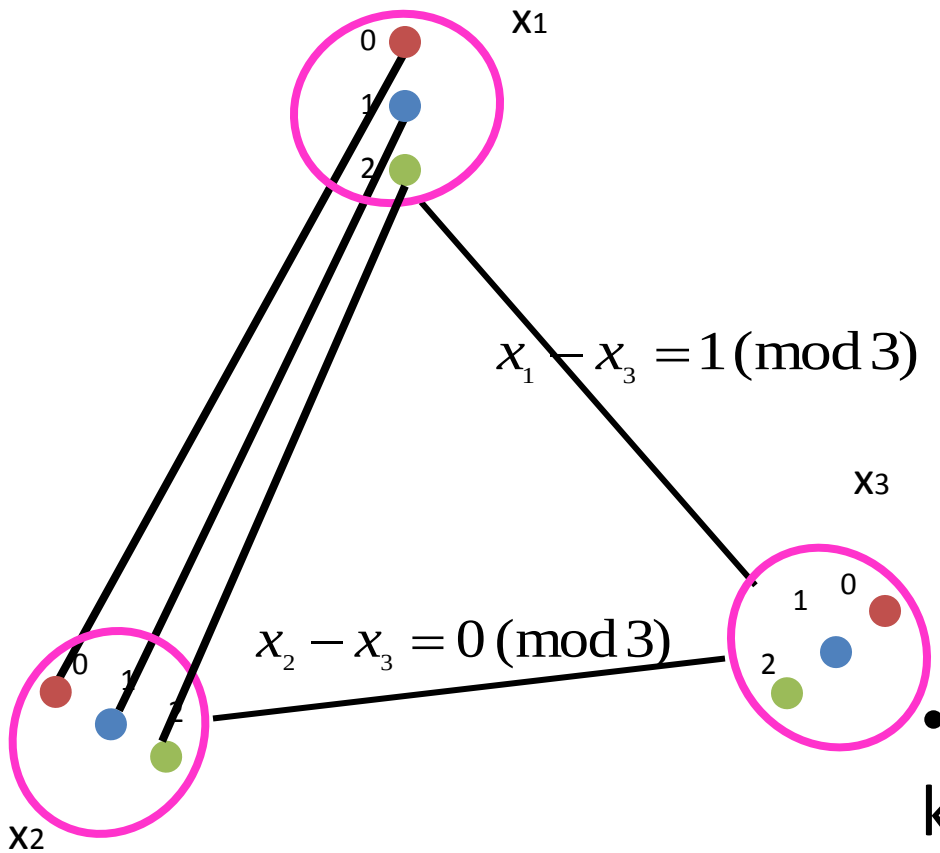

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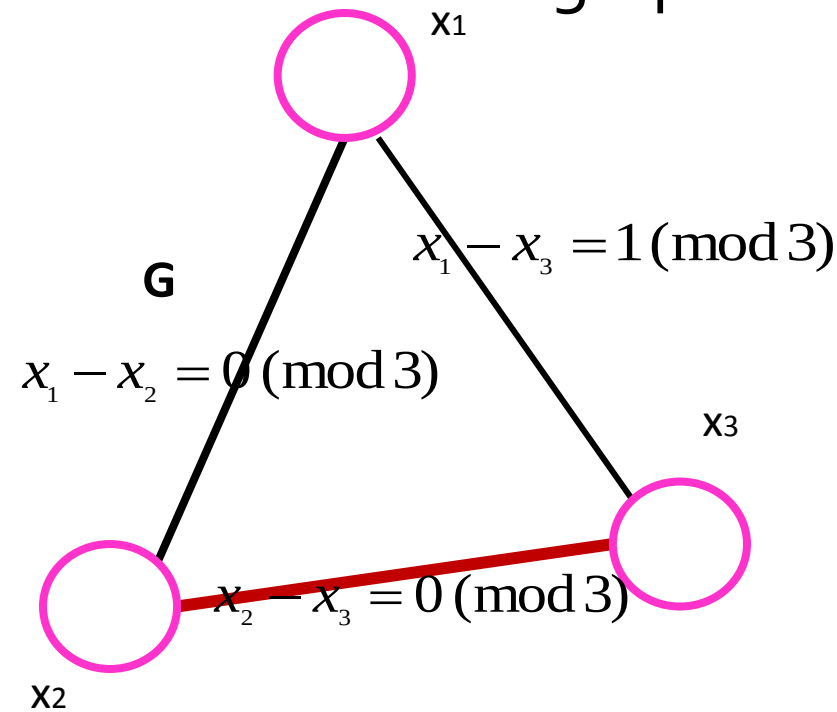

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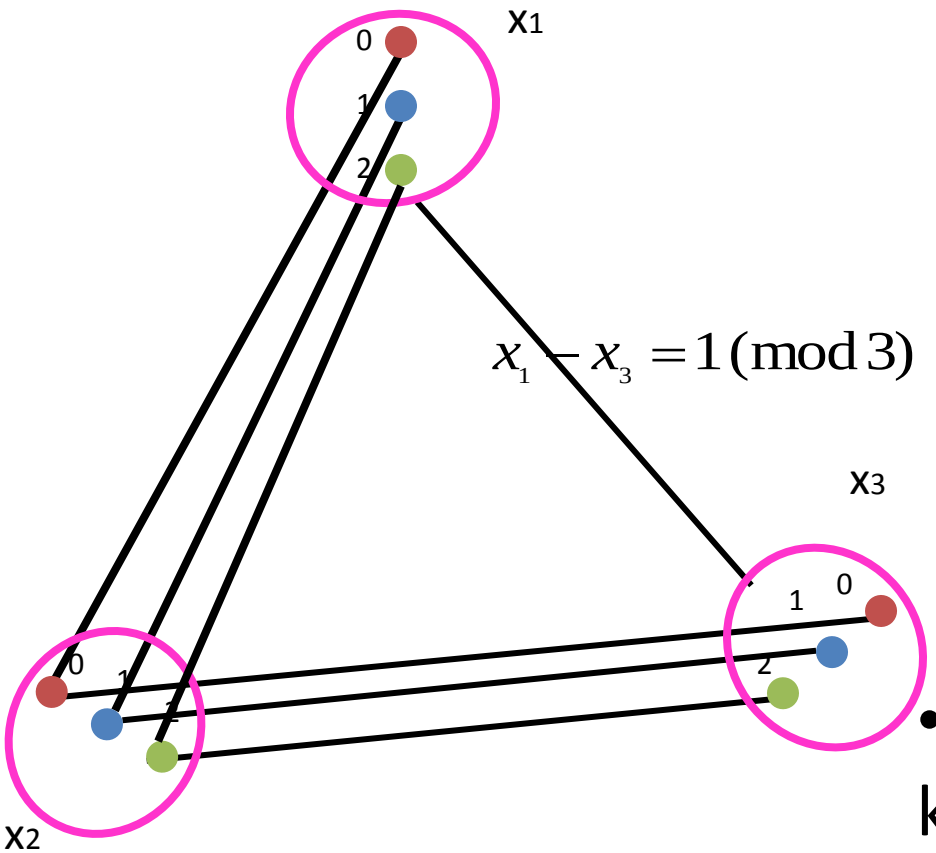

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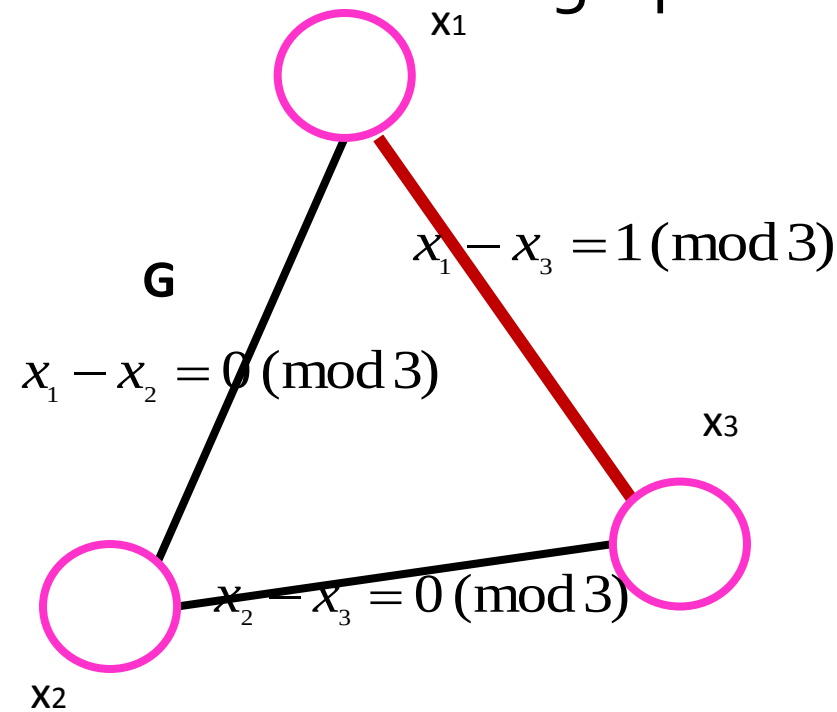

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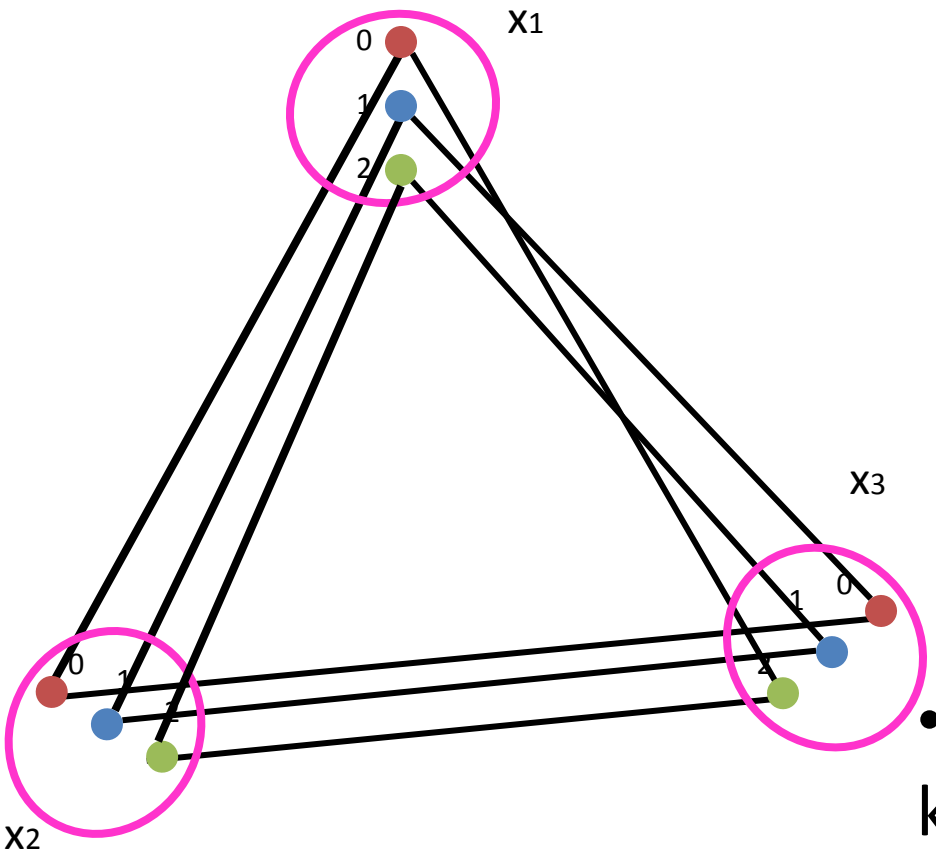

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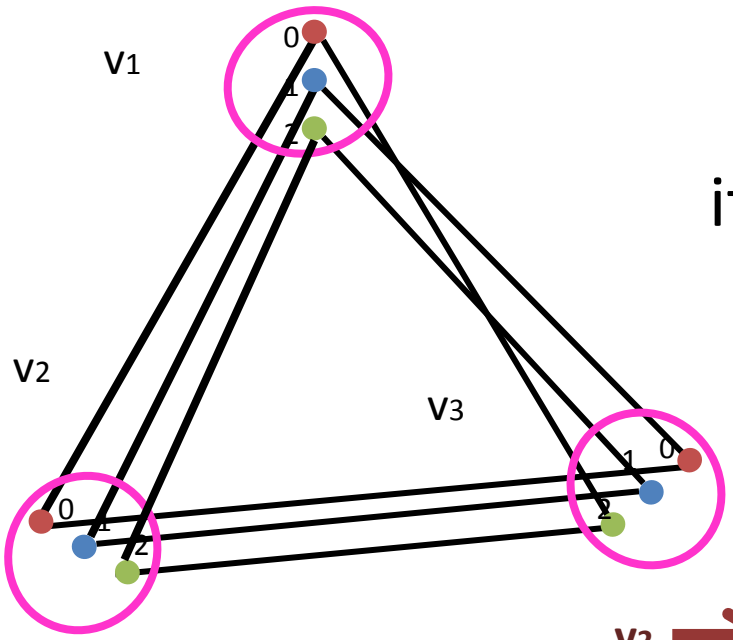
More Graph Theory: The Label-Extended Graph



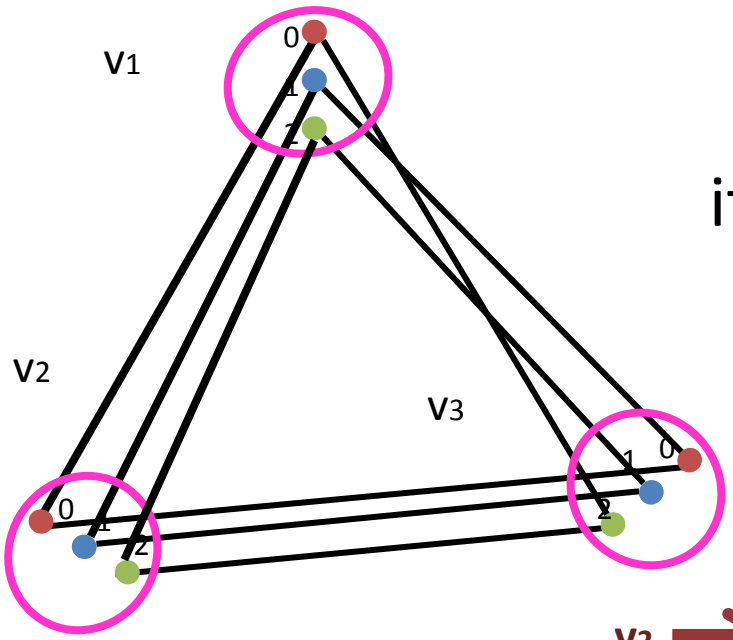
GRAPH THEORY?

it's a graph, it has adjacency matrix!

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
2 & 0 & 0 & 1 \\
\hline
\end{array}
$$

M has each non – zero entry replaced by a block corresponding to the permutation on edge

More Graph Theory: The Label-Extended Graph



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$$
\begin{array}{c|ccc}
 & v_1 & & \\
 & 0 & 1 & 2 \\
\hline
0 & 1 & 0 & 0 \\
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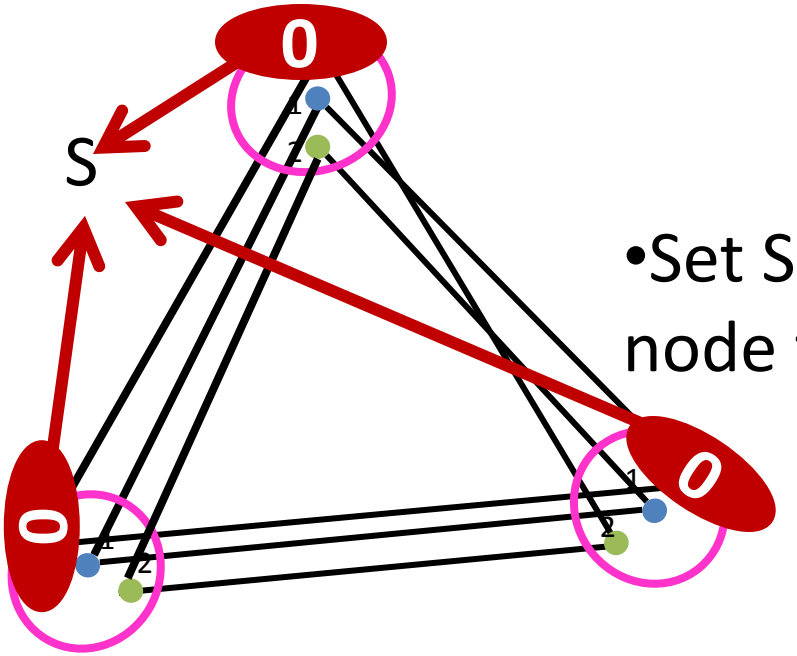
\end{array}
$$

v_2

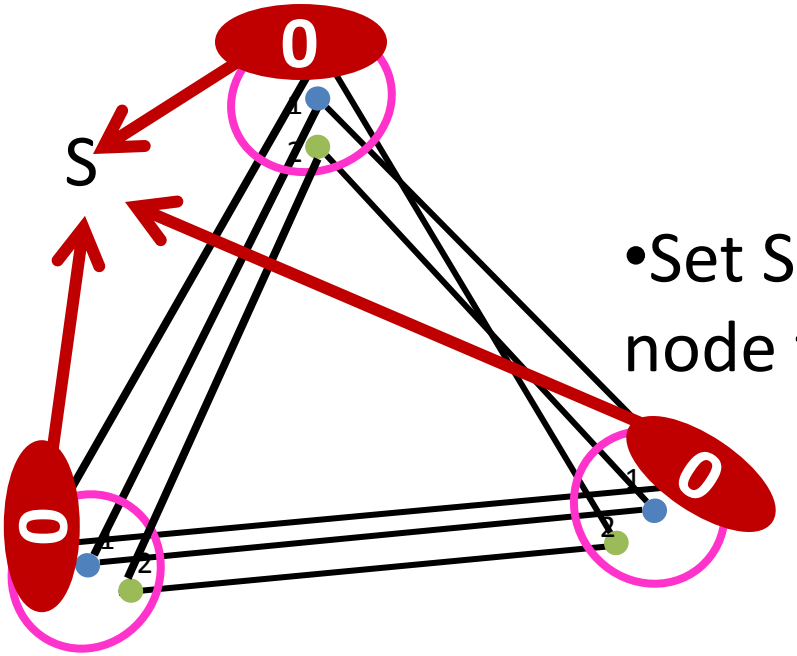
"Old eigenvalues" of original graph are still eigenvalues. What other eigenvalues are there?

From Labelings to Spectra



- Set S that contains exactly one "small" node from each node group = labeling

From Labelings to Spectra

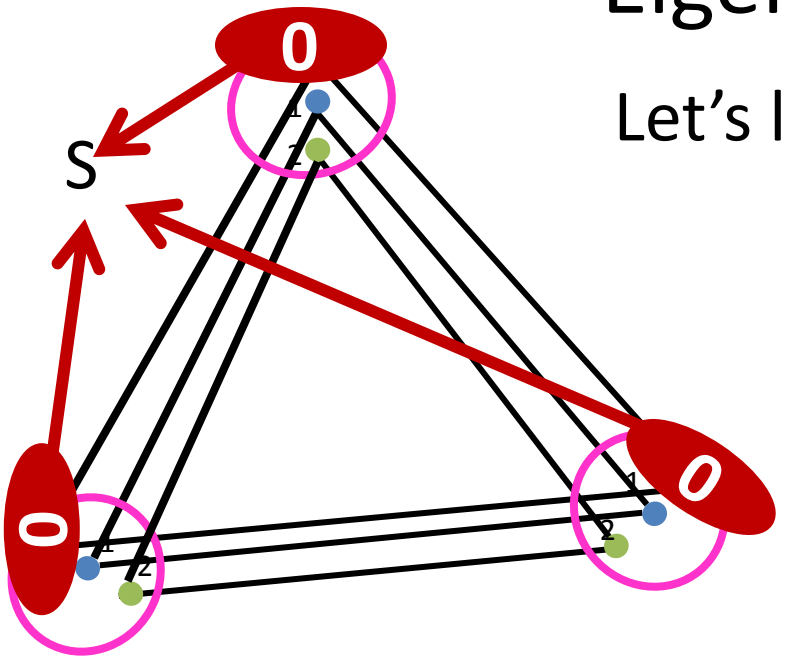


- Set S that contains exactly one "small" node from each node group = labeling

- Corresponds to a "characteristic vector" (cut vector)

$$\chi_{(0,0,0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Perfect Satisfiability and Largest Eigenvalues



Let's look at a perfectly satisfiable game for intuition…

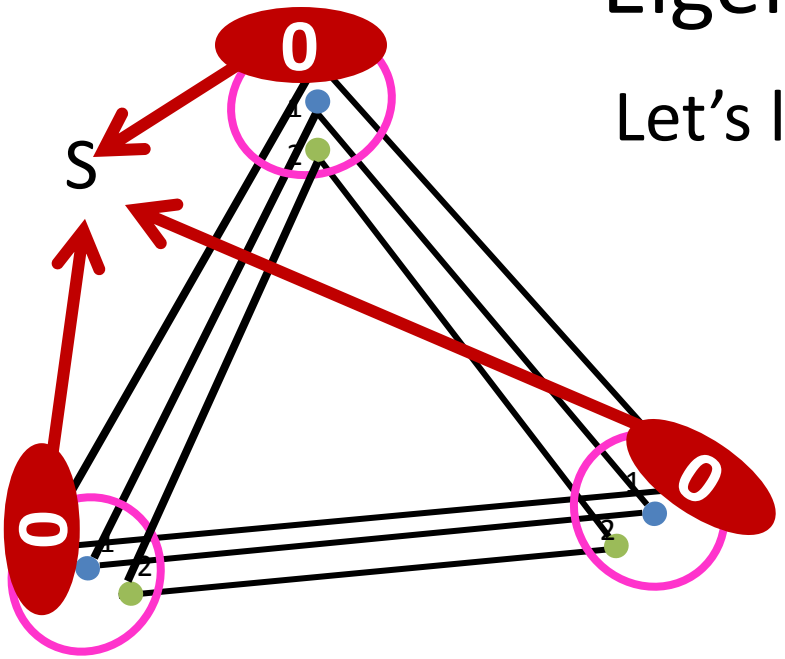
$$\chi_{(0,0,0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

- Corresponds to a perfect labeling, cut that cuts no edges

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

- Is eigenvector of M with eigenvalue d

Perfect Satisfiability and Largest Eigenvalues



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- Corresponds to a perfect labeling, cut that cuts no edges

$$L = dI - M$$

- Is eigenvector of L with eigenvalue 0 (k connected components)

On the Opposite Side: Very Unsatisfiable Instances

We similarly expect that very unsatisfiable Instances on “well-connected” graphs will have second Adjacency matrix eigenvalue is very small (far from d), or the second Laplacian eigenvalue is very large, far from 0.

“Well-connected” for us will mean that, if graph is d-regular, then the second Adjacency Matrix eigenvalue is $\sim O\left(\sqrt{d}\right)$ and the second Laplacian eigenvalue is $\sim d - \Omega\left(\sqrt{d}\right)$

SDP Relaxation

- Introduce an "indicator vector" u_i for each vertex u and color i

$$u_i = \begin{cases} e, \text{if } u \text{ is colored with } i \\ 0, \text{otherwise} \end{cases}$$

- Number of unsatisfied constraints equals

$$\sum_{uv \in E} \sum_{i} \left\| u_i - v_{\pi_{uv}(i)} \right\|^2$$

SDP Relaxation

- Introduce an "indicator vector" u_i for each vertex u and color i.

- Minimize : $$\sum_{uv \in E} \sum_{i} \left\| u_i - v_{\pi_{uv}(i)} \right\|^2$$

  Subject to: $\|u_1\|^2 + \ldots + \|u_k\|^2 = 1$ for all u

  $$\langle u_i | u_i \rangle = 0, \text{for all } u, i \neq j$$

- We next use dual to bound the primal (blackboard). Led to UG are easy on expanders