Expander Codes
Today

- Graph approximations, part 2. Bipartite expanders as approximations of the bipartite complete graph
- Quasi-random properties of bipartite expanders, expander mixing lemma
- Building expander codes
- Encoding, minimum distance, decoding
Expander Graph Refresher

- We defined expander graphs to be $d$-regular graphs whose adjacency matrix eigenvalues satisfy $|\alpha_i| \leq \epsilon d$ for $i > 1$, and some small $\epsilon$.
- We saw that such graphs are very good approximations of the complete graph.
Let $G$ be a $d$-regular graph whose adjacency eigenvalues satisfy $|\alpha_i| \leq \epsilon d$.

As its Laplacian eigenvalues satisfy $\lambda_i = d - \alpha_i$, all non-zero eigenvalues are between $(1 - \epsilon)d$ and $(1 + \epsilon)d$.

Let $H=(d/n)K_n$, so $x^T L_H x = dx^T x$

We showed that $G$ is an $\epsilon -$ approximation of $H$

$$(1 - \epsilon)x^T L_H x \leq x^T L_G x \leq (1 + \epsilon)x^T L_H x$$
Bipartite Expander Graphs

- Define them as $d$-regular good approximations of (some multiple of) the complete bipartite graph $H = \frac{d}{n}K_{n,n}$:

\[
(1 - \epsilon)x^T L_H x \leq x^T L_G x \leq (1 + \epsilon)x^T L_H x
\]

\[
\Rightarrow (1 - \epsilon)H \preceq G \preceq (1 + \epsilon)H
\]
The Spectrum of Bipartite Expanders

- Eigenvalues of Laplacian of $H=\frac{d}{n}K_{n,n}$ are $\lambda_1 = 0, \lambda_{2n} = 2d, \lambda_i = d, 0 < i < 2n$

- $(1 - \epsilon)x^T L_H x \leq x^T L_G x \leq (1 + \epsilon)x^T L_H x$
  Says we want $d$-regular graph $G$ that has evals $\lambda_1 = 0, \lambda_{2n} = 2d$, $\lambda_i \in ((1 - \epsilon)d, (1 + \epsilon)d), 0 < i < 2n$
Construction of Bipartite Expanders

- **Definition (Double Cover).** Let $G=(V,E)$ be a graph. The double-cover of $G$ is the graph with vertex set $V \times \{0,1\}$ and edges $((u,0),(v,1))$ for $(u,v)$ in $E$.

Cycle on 4 vertices and its double-cover
Construction of Bipartite Expanders

- Proposition (Bipartite Expanders as Double Cover of Expanders). Let $G=(V,E)$ be a $d$-regular graph and $H$ its double cover. For every eigenvalue $\lambda_i$ of the Laplacian of $G$, $H$ has a pair of eigenvalues $\lambda_i, 2d-\lambda_i$.

- We construct bipartite expander from double cover of $d$-regular expander
Expander Mixing Lemma, contd.

- **Theorem** Let $G=(U \cup V, E)$ a $d$-regular bipartite graph that $\epsilon$-approximates the graph $\frac{d}{n} K_{n,n}$. Then, for every $S \subseteq U$, $T \subseteq V$

  \[ ||E(S, T) - d \frac{|S||T|}{n} | \leq \epsilon d \sqrt{|S||T|} \]
Average Degree.

- **Theorem** Let $|S| = \sigma n$, $|T| = \tau n$. The average degree of vertices in $G(SUT)$ is at most $\frac{2d\sigma\tau}{\sigma + \tau} + \epsilon d$.
Our construction of error-correcting codes will require two ingredients:

- A $d$-regular bipartite expander $G$ on $2n$ vertices
- A linear error-correcting code $C_0$ of length $d$ (e.g., Hamming)

We combine the two to construct code of length $dn$. Think of $C_0$ as small code that drives the construction ($d$ small as $n \to \infty$)
Building Expander Codes

- Associate one bit with each edge of G.
- G has $dn$ edges $\rightarrow$ code of $2n$ bits $y_1, \ldots, y_{dn}$
- Describe the code by listing the linear constraints its codewords must satisfy: for each vertex, we require that the $d$ bits corresponding to its attached $d$ edges are a codeword in $C_0$. 
Building Expander Codes

- If \( r_0 \) is the rate of \( C_0 \) then the space of codewords has dimension \( d r_0 \).
- Since \( C_0 \) is linear, its codewords satisfy some set of \( d(1 - r_0) \) linear equations.
- As there are \( 2n \) vertices in \( G \), the constraints imposed by each vertex impose \( 2nd(1 - r_0) \) linear constraints on the \( dn \) bits.
- Thus, the vector space of codewords that satisfies all of those constraints has dimension at least \( dn - 2nd(1 - r_0) = dn(2r_0 - 1) \).
- Code we constructed has rate at least \( r = 2r_0 - 1 \). Non-zero as long as \( r_0 > 1/2 \).
Building Expander Codes

- Encoding? Parity Check matrix, Generator Matrix.
- Minimum Distance?
- Decoding?