



# CS 598: Spectral Graph Theory. Lecture 11

Expander Graphs and  
Their Properties

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# Today

- Graph approximations. Expanders as approximations of the complete graph
- Quasi-random properties of expanders, expander mixing lemma
- Vertex expansion
- How well can we approximate the complete graph?

# Expander Graphs via PRGs

- Our PRG from last lecture uses a random walk on a  $d$ -regular  $G$  with constant degree.
- We wanted  $G$  to be an expander in the following sense: If  $A_G$  is  $G$ 's adjacency matrix and  $d = \alpha_1 > \alpha_2 \geq \dots \geq \alpha_n$  its eigenvalues then we require that  $\frac{|\alpha_i|}{d} \leq \frac{1}{10}$ .
- This allowed us to show that walk cannot stay in a small set for a long time.

# Expander Graphs

- Generally, we define expander graphs to be  $d$ -regular graphs whose adjacency matrix eigenvalues satisfy

$$|\alpha_i| \leq \epsilon d$$

for  $i > 1$ , and some small  $\epsilon$ .

- We will next see that such graphs are very good approximations of the complete graph.

# Graphic Inequalities

- In Lecture 4, we defined

$$G \succcurlyeq H \quad \text{if} \quad L_G \succcurlyeq L_H$$

Or, equivalently,  $v^T L_G v \geq v^T L_H v$  for all  $v$

- We say that  $G$  is an  $\epsilon$  – approximation of  $H$  if

$$(1 - \epsilon)H \preccurlyeq G \preccurlyeq (1 + \epsilon)H$$

# Approximations of the Complete Graph

- Let  $G$  be a  $d$ -regular graph whose adjacency eigenvalues satisfy  $|\alpha_i| \leq \epsilon d$ .
- As its Laplacian eigenvalues satisfy  $\lambda_i = d - \alpha_i$ , all non-zero eigenvalues are between  $(1 - \epsilon)d$  and  $(1 + \epsilon)d$ .
- This means that for all  $x$  orthogonal to the all-one's vector

$$(1 - \epsilon)dx^T x \leq x^T L_G x \leq (1 + \epsilon)dx^T x$$

# Approximations of the Complete Graph

- For the complete graph  $K_n$ , we know all  $x$  orthogonal to the all one's vector satisfy  $x^T L_{K_n} x = nx^T x$
- Let  $H=(d/n)K_n$ , so  $x^T L_H x = dx^T x$
- This means that  $G$  is an  $\epsilon$  - approximation of  $H$   
 $(1 - \epsilon)dx^T x \leq x^T L_G x \leq (1 + \epsilon)dx^T x \Rightarrow$   
 $(1 - \epsilon)H \leq G \leq (1 + \epsilon)H \Rightarrow$   
 $-\epsilon H \leq G - H \leq \epsilon H \Rightarrow \|L_H - L_G\| \leq \epsilon d$

# Quasi-Random Properties of Expander Graphs

- Expanders act like random graphs in many ways.
- We saw last time that with random walk on expander, we can boost the error probability like we could do with random walk on a random graph (or truly random stings, Chernoff bound)
- In fact, a random  $d$ -regular graph is expander w.h.p.



# Quasi-Random Properties of Expander Graphs

- All sets of vertices in expander graph act like random sets of vertices.
- To see that, consider creating a random set  $S \subseteq V$  by including every vertex in  $S$  independently w.p.  $a$ .
- For every edge  $(u,v)$  the probability that each end point is in  $S$  is  $a$ . Probability that both end points are in  $S$  is  $a^2$ .
- So, we expect  $a^2$  fraction of the edges to go between vertices in  $S$ .
- We show that this is true for all sufficiently large sets in an expander.

# Quasi-Random Properties of Expander Graphs: EML

- We show something stronger (expander mixing lemma), for two sets  $S$  and  $T$ .
- Include each vertex in  $S$  w.p.  $a$  and each vertex in  $T$  w.p.  $b$ . We allow vertices to belong to both  $S$  and  $T$ . We expect that for  $ab$  fraction of ordered pairs  $(u,v)$  we have  $u$  in  $S$  and  $v$  in  $T$ .

# Expander Mixing Lemma

- For graph  $G=(V,E)$  define the ordered set of pairs

$$\overrightarrow{E(S,T)} = \{(u,v): u \in S, v \in T, (u,v) \in E\}$$

- When  $S, T$  disjoint  $|\overrightarrow{E(S,T)}|$  is the number of edges between  $S$  and  $T$ .
- $|\overrightarrow{E(S,S)}|$  counts every edge inside  $S$  twice.

# Expander Mixing Lemma

- **Theorem** (Beigel, Margulis, Spielman'93, Alon, Chung '88)

Let  $G=(V,E)$  a  $d$ -regular graph that  $\epsilon$ -approximates the graph  $\frac{d}{n}K_n$ . Then, for every  $S \subseteq V$ ,  $T \subseteq V$  with  $|S|=an$ ,  $|T|=bn$

$$\left| \overrightarrow{|E(S,T)|} - d \frac{|S||T|}{n} \right| \leq \epsilon d \sqrt{|S||T|} \Rightarrow$$
$$\left| \overrightarrow{|E(S,T)|} - dabn \right| \leq \epsilon dn \sqrt{ab}$$

# Expander Mixing Lemma(stronger version)

- **Theorem** (Beigel, Margulis, Spielman'93, Alon, Chung '88)

Let  $G=(V,E)$  a  $d$ -regular graph that  $\epsilon$ -approximates the graph  $\frac{d}{n}K_n$ . Then, for every  $S \subseteq V$ ,  $T \subseteq V$  with  $|S|=an$ ,  $|T|=bn$

$$|\overrightarrow{|E(S,T)|} - dabn| \leq \epsilon dn \sqrt{(a - a^2)(b - b^2)}$$

# Vertex Expansion

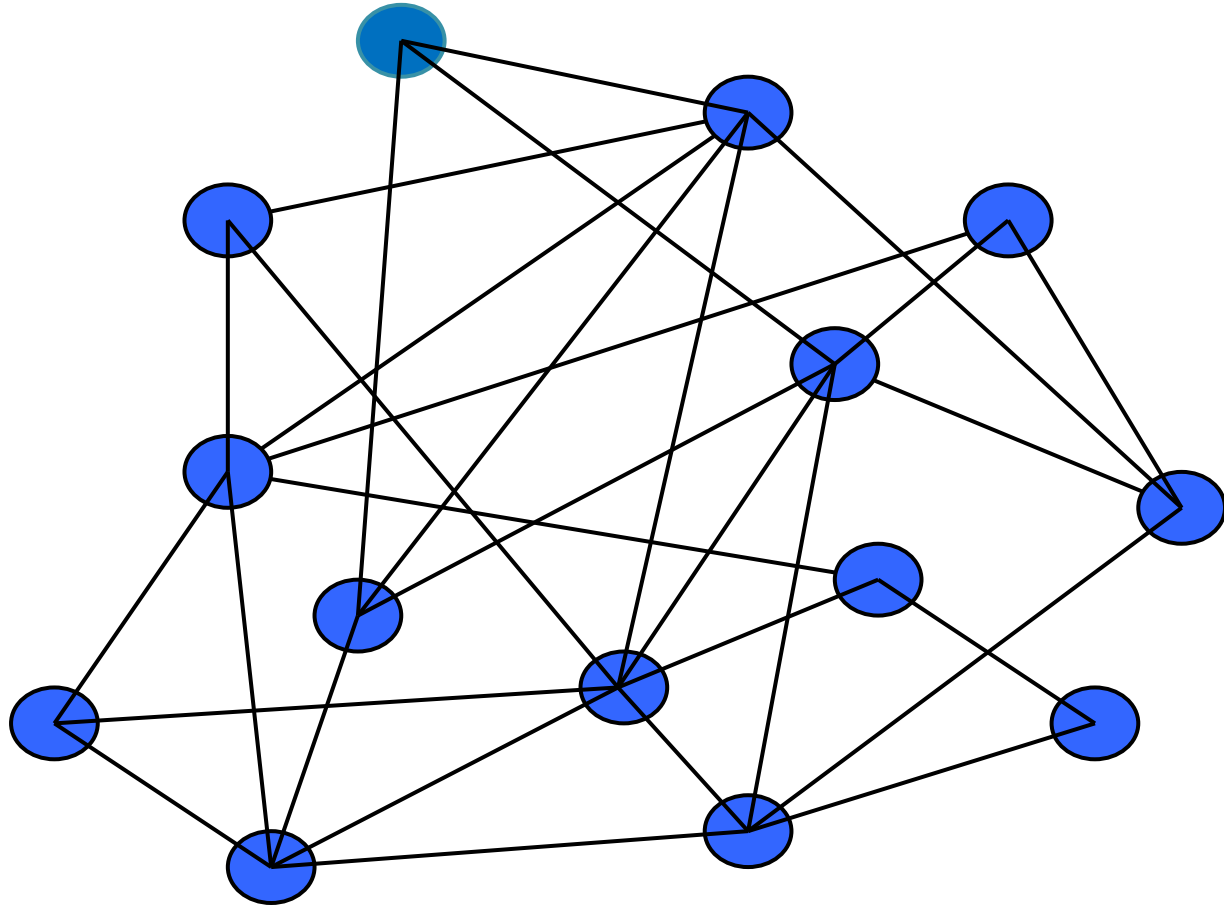
- The name Expander came from the fact that small sets of vertices have unusually large numbers of neighbors. For a subset  $S$  of vertices, let  $N(S)$  denote the set of vertices that are neighbors of vertices in  $S$ .
- **Theorem** (Tanner '84).

Let  $G=(V,E)$  a  $d$ -regular graph that  $\epsilon$ -approximates the graph  $\frac{d}{n} K_n$ . Then, for every  $S \subseteq V$ ,  $|S|=an$ ,

$$|N(S)| \geq \frac{|S|}{\epsilon^2(1-a) + a}$$

# How Well Can a Graph Approximate the Complete Graph?

***G***



# How Well Can a Graph Approximate the Complete Graph?

- Apply Tanner for  $S=\{v\}$ , a single vertex.
- Since  $v$  has exactly  $d$  neighbors, we find that 
$$\epsilon^2 \left(1 - \frac{1}{n}\right) + \frac{1}{n} \geq \frac{1}{d} \implies \epsilon \gtrsim \frac{1}{\sqrt{d}}$$
- But how small can it get?
- Ramanujan graphs [Margulis, LPS '88] give  $\implies$  
$$\epsilon \leq \frac{2\sqrt{d-1}}{d}$$
- We will see that if  $d$  fixes as  $n$  grows, the bound is tight in the limit.