CS 598: Spectral Graph Theory. Lecture 1

The Laplacian

Alexandra Kolla
Administrativia

- Email, office, office hours, course website, scribe notes, homeworks, projects, prerequisites.
- Course goals.
Course Overview

Rough layout. The topics covered will also depend on student interest. Some of the lectures will be in the form of presentations on some advanced topics that we didn’t cover in class.

- Graphs, matrices and their spectra (4 Lectures)
- Topics on the second eigenvalue (8 Lectures)
- Topics on higher eigenvalues (3 Lectures, maybe more, depending on interest)
- Topics on all eigenvalues-graph approximations (3-4 Lectures)
Graphs, Matrices and their Spectra

- Adjacency matrix, diffusion operator, Laplacian.
- Eigenvalues and eigenvectors of graphs, examples.
- Properties of the Laplacian, properties of adjacency matrix and their relations.
- Courant-Fischer, Perron-Frobenius, nodal domains.
- Eigenvalue bounding techniques, examples.
Topics on the Second Eigenvalue

- Edge expansion, graph cutting, Cheeger’s inequality.
- Semidefinite programming, duality and connections with the second eigenvalue.
- Second eigenvalue for planar graphs.
- Expanders: existence, constructions and applications.
Topics on Higher Eigenvalues

- Approximation algorithm for MAXCUT using the last eigenvalue.
- Small set expansion and higher eigenvalues, Cheeger-like inequalities.
- The Unique Games Conjecture and approximation algorithms using higher eigenvalues.
- Semidefinite programming hierarchies and higher spectra (maybe).
Topics on All Eigenvalues-Graph Approximations

- Various graph approximations, sparsification and applications.
- Spectral sparsifiers with effective resistances.
- Solving Laplacian systems of linear equations with preconditioning, ultrasparsifiers.
In the next few minutes:

Why spectral graph theory is both natural and magical
Representing Graphs

Obviously, we can represent a graph with an nxn matrix

Adjacency matrix

\[ A_{ij} = \begin{cases} 
    w_{ij} & \text{weight of edge (i, j)} \\
    0 & \text{if no edge between i, j}
\end{cases} \]
What is not so obvious, is that once we have matrix representation view graph as linear operator.

- Can be used to multiply vectors.
- Vectors that don’t rotate but just scale = eigenvectors.
- Scaling factor = eigenvalue

\[ Ax = \mu x \]

Amazing how this point of view gives information about graph.

Obviously, we can represent a graph with an nxn matrix.
"Listen" to the Graph

List of eigenvalues
\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}: graph SPECTRUM

Adjacency matrix

Eigenvalues reveal **global** graph properties
not apparent from edge structure

A drum:

Hear shape of the drum
“Listen” to the Graph

List of eigenvalues
\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n \}: graph SPECTRUM

Eigenvalues reveal \textit{global} graph properties
not apparent from edge structure

Hear \textit{shape of the drum} 
Its sound:
“Listen” to the Graph

List of eigenvalues
\{\mu_1 \geq \mu_2 \geq ... \geq \mu_n \}: graph SPECTRUM

Eigenvalues reveal **global** graph properties
not apparent from edge structure

Hear shape of the drum

Its sound (eigenfrequencies):

Adjacency matrix

\[
A = \begin{pmatrix}
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
\end{pmatrix}
\]
“Listen” to the Graph

List of eigenvalues
\{\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n\}: graph SPECTRUM

Eigenvalues reveal global graph properties not apparent from edge structure

If graph was a drum, spectrum would be its sound
Eigenvectors are Functions on Graph

\[ v \in \mathbb{R}^n, \quad v : V \rightarrow \mathbb{R} \]

\[ Av = \mu v \]

\[ v(i) = \text{value at node } i \]
Eigenvectors are Functions on Graph

\[ v \in \mathbb{R}^n, \quad v : V \rightarrow \mathbb{R} \quad A v = \mu v \]

\[ v(i) = \text{value at node } i \quad \text{different shade of grey} \]
So, let’s See the Eigenvectors

* Slides from Dan Spielman
The second eigenvector
Third Eigenvector

* Slides from Dan Spielman
Fourth Eigenvector

* Slides from Dan Spielman
Another view: the Laplacian

We can also view graph as Laplacian

\[ L = D - A \]

where \( D \) is diagonal matrix of degrees

\[ L_{ij} = \begin{cases} 
  d_i & \text{if } i = j \\
  -w_{ij} & \text{if } (i, j) \text{ edge} \\
  0 & \text{otherwise}
\end{cases} \]
The Laplacian: Fast Facts

\[ L \mathbf{1} = 0 \]

so, zero is an eigenvalue
\( \mathbf{1} \) an eigenvector

\[ 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \]

SPECTRUM of the Laplacian

\[ \lambda_2 > 0 \iff \text{Graph CONNECTED} \]

\( \lambda_2 \) also “algebraic connectivity”

The further from 0, the more connected
Cuts and Algebraic Connectivity

Cuts in a graph:

\[ \text{cut}(S, S') = \frac{E(S, S')}{|S|}, |S| \leq n/2 \]

Graph not well-connected when “easily” cut in two pieces
Cuts and Algebraic Connectivity

Sparsest Cut:
\[ h(G) = \min_{S:|S| \leq n/2} \frac{E(S, \bar{S})}{|S|} \]

Graph not well-connected when “easily” cut in two pieces

Would like to know Sparsest Cut but NP hard to find

How does algebraic connectivity relate to standard connectivity?

**Theorem** (Cheeger-Alon-Milman):
\[ \lambda_2 \leq h(G) \leq \sqrt{2d_{\text{max}}} \sqrt{\lambda_2} \]
Cuts and Algebraic Connectivity

Sparsest Cut:

\[ h(G) = \min_{S : |S| \leq n/2} \frac{E(S, \bar{S})}{|S|} \]

Graph not well-connected when “easily” cut in two pieces

Would like to know Sparsest Cut but NP hard to find

How does algebraic connectivity relate to standard connectivity?

Algebraic connectivity large ↔ Graph well-connected
Graphs with no Small Cuts

Certain graphs have no small cuts: **Expanders**

Very useful for applications

- Constructing robust networks
- Routing

Some far less obvious ones:
- Probability amplification
- Error correcting codes

would like to **build** expanders
Graphs with no Small Cuts

Certain graphs have no small cuts: **Expanders**

Very useful for applications

- Constructing robust networks
-Routing

Some far less obvious ones:
- Probability amplification
- Error correcting codes…

Two equivalent ways to understand them helped find them

- No small cuts
- Large Algebraic Connectivity
Today

- More on evectors and evals
- The Laplacian, revisited
- Properties of Laplacian spectra, PSD matrices.
- Spectra of common graphs.
- Start bounding Laplacian evals
A Remark on Notation

For convenience, we will often use the bra-ket notation for vectors:

- We denote vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ with a “bra”: $|v\rangle$

- We denote the transpose vector $\mathbf{v}^T = (v_1 \ldots v_n)$ with a “ket”: $\langle v |$

- We denote the inner product $\mathbf{v}^T \mathbf{u}$ between two vectors $\mathbf{v}$ and $\mathbf{u}$ with a “braket”: $\langle v | u \rangle = \langle v, u \rangle$
E vectors and E values

- Vector $v$ is an eigenvector of matrix $A$ with eigenvalue $\mu$ if $Av = \mu v$.
- We are interested (almost always) in symmetric matrices, for which the following special properties hold:
  
  - If $v_1, v_2$ are eigenvectors of $A$ with eigenvalues $\mu_1, \mu_2$ and $\mu_1 \neq \mu_2$, then $v_1$ is orthogonal to $v_2$. (Proof)
  
  - If $v_1, v_2$ are eigenvectors of $A$ with the same eigenvalue $\mu$, then $v_1 + v_2$ is as well. The multiplicity of eigenvalue $\mu$ is the dimension of the space of eigenvectors with eigenvalue $\mu$.
  
  - Every $n$-by-$n$ symmetric matrix has $n$ eigenvalues $\{\mu_1 \leq \cdots \leq \mu_n\}$ counting multiplicities, and an orthonormal basis of corresponding eigenvectors $\{v_1, \ldots, v_n\}$, so that $Av_i = \mu_i v_i$
  
  - If we let $V$ be the matrix whose $i$-th column is $v_i$, and $M$ the diagonal matrix whose $i$-th diagonal is $\mu_i$, we can compactly write $AV = VM$. Multiplying by $V^T$ on the right, we obtain the eigendecomposition of $A$:

$$A = AV V^T = VM V^T = \sum_i \mu_i v_i v_i^T$$
The Laplacian: Definition Refresher

\[ L_G = \begin{cases} 
  d_i & \text{if } i = j \\
  -1 & \text{if } (i, j) \text{ edge} \\
  0 & \text{otherwise}
\end{cases} \]

Where \( d_i \) is the degree of \( i \)-th vertex.
For convenience, we have unweighted graphs

- \( D_G = \text{Diagonal matrix of degrees} \)
- \( A_G = \text{Adjacency matrix of the graph} \)
- \( L_G = D_G - A_G \)
The Laplacian: Properties Refresher

• The constant vector $1$ is an eigenvector with eigenvalue zero.
  \[ L_G \vec{1} = 0 \]

• Has n eigenvalues (spectrum) \[ 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \]

• Second eigenvalue is called “algebraic connectivity”. G is connected if and only if $\lambda_2 > 0$.

• We will see the further away from zero, the more connected G is.
Redefining the Laplacian

- Let $L_e$ be the Laplacian of the graph on $n$ vertices consisting of just one edge $e=(u,v)$.

\[
L_e(i, j) = \begin{cases} 
1 & \text{if } i = j, i \in u, v \\
-1 & \text{if } i = u, j = v, \text{ or vice versa} \\
0 & \text{otherwise}
\end{cases}
\]

- For a graph $G$ with edge set $E$ we now define

\[
L_G = \sum_{e \in E} L_e
\]

- Many elementary properties of the Laplacian now follow from this definition as we will see next (prove facts for one edge and then add).
Laplacian of an edge, contd.

\[ L_e = \begin{bmatrix} u & v \\ 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes [zeros] \]

\[
\left( \begin{array}{cc}
1 & -1 \\
-1 & 1 \\
\end{array} \right) = \left( \begin{array}{cc}
1 & -1 \\
-1 & 1 \\
\end{array} \right) \cdot \begin{bmatrix} 1 - 1 \end{bmatrix} = 2 \left( \begin{array}{c}
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} \\
\end{array} \right) \left( \begin{array}{c}
\frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} \\
\end{array} \right)
\]

- Since eigenvalues are zero and 2, we see that \( L_e \) is P.S.D. Moreover,

\[
x^T L_e x = (x_1 x_2) \left( \begin{array}{cc}
1 & -1 \\
-1 & 1 \\
\end{array} \right) \begin{bmatrix} x_1 \\
-1 \\
x_2 \end{bmatrix} = (x_1 - x_2)^2
\]
Review of Positive Semidefiniteness

- **Definition:** A symmetric matrix $M$ is positive semidefinite (PSD) if:

  $$x^T M x \geq 0 \, \forall \, x \in \mathbb{R}^n$$

  Positive definite (PD) if inequality is strict for all $x \neq 0$.

- PSD iff all eigenvalues are non-negative (exercise.)

- PSD iff $M$ can be written as $M = A^T A$, where $A$ can be $n$-by-$k$ (not necessarily symmetric) and is not unique.
  
  Proof: see blackboard
More Properties of Laplacian

From the definition using edge sums, we get:

• **(PSD-ness)** The Laplacian of any graph is PSD.

\[ x^T L_G x = x^T \left( \sum_{e \in E} L_e \right) x = \sum_{e \in E} x^T L_e x = \sum_{(i,j) \in E} (x_i - x_j)^2 \]

• **(Connectivity)** G is connected iff \( \lambda_2 \) positive or alternatively, the null space of the Laplacian of G is 1-dimensional and spanned by the vector \( \mathbf{1} \). (Proof on blackboard)

• **Corollary:** The multiplicity of zero as an eigenvalue equals the number of connected components of the graph.
More Properties of Laplacian

- **(Edge union)** If $G$ and $H$ are two graphs on the same vertex set, with disjoint edge set then

  \[ L_{G \cup H} = L_G + L_H \text{ (additivity)} \]

- If a vertex is isolated, the corresponding row and column of Laplacian are zero

- **(Disjoint union)** Together these imply that for the disjoint union of graphs $G$ and $H$

  \[ L_{G \upuparrows H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix} \]
More Properties of Laplacian

- **(Edge union)** If $G$ and $H$ are two graphs on the same vertex set, with disjoint edge set then
  \[ L_{G\cup H} = L_G + L_H \text{ (additivity)} \]

- If a vertex is isolated, the corresponding row and column of Laplacian are zero

- **(Disjoint union)** Together these imply that for the disjoint union of graphs $G$ and $H$
  \[ L_{G \boxplus H} = L_G \oplus L_H = \begin{pmatrix} L_G & 0 \\ 0 & L_H \end{pmatrix} \]

- **(Disjoint union spectrum)** If $L_G$ has evectors $v_1, \ldots, v_n$ with evalues $\lambda_1, \ldots, \lambda_n$ and $L_H$ has evectors $w_1, \ldots, w_n$ with evalues $\mu_1, \ldots, \mu_n$ then $L_{G \boxplus H}$ has evectors $v_1 \oplus 0, \ldots, v_n \oplus 0, 0 \oplus w_1, \ldots, 0 \oplus w_n$ with evalues $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n$. 
The Incidence Matrix: Factoring the Laplacian

- We can factor $L = V^TV$ using eigenvectors but also exists nicer factorization
- Define the incidence matrix $B$ to be the $m$-by-$n$ matrix
  \[ B(e, v) = \begin{cases} 
  1, & \text{if } e = (v, w) \text{ and } v < w \\
  -1, & \text{if } e = (v, w) \text{ and } w < v \\
  0, & \text{otherwise} 
  \end{cases} \]
- Example of incidence matrix
  \[
  B = \begin{pmatrix}
  1 & -1 & 0 \\
  1 & 0 & -1 \\
  
  \end{pmatrix}
  \]
  \[
  L = \begin{pmatrix}
  1 & -1 & 0 \\
  -1 & 2 & -1 \\
  0 & -1 & 1 
  \end{pmatrix}
  \]
- Claim: $L = B^TB$ (exercise)
- Gives another proof that $L$ is PSD.
Spectra of Some Common Graphs

- The complete graph $K_n$ on $n$ vertices with edge set $\{(u, v): u \neq v\}$
- The path graph $P_n$ on $n$ vertices with edge set $\{(u, u + 1): 0 \leq u < n\}$
- The ring graph $R_n$ on $n$ vertices with edge set $\{(u, u + 1): 0 \leq u < n\} \cup (0, n - 1)$
- The grid graph $G_{nxm}$ on $nxm$ vertices with edges from node $(u_1, u_2)$ to nodes that differ by one in just one coordinate
- Product graphs in general
The Complete Graph

\[ K_n: \{(u, v): u \neq v\} \]

- The Laplacian of \( K_n \) has eigenvalue zero with multiplicity 1 (since it is connected) and \( n \) with multiplicity \( n-1 \).

- Proof: see blackboard
The Ring Graph

$R_n: \{(u, u + 1): 0 \leq u < n\} \cup (0, n - 1)$

- The Laplacian of $R_n$ has eigenvectors
  
  $x_k(u) = \sin\left(\frac{2\pi ku}{n}\right)$ and
  $y_k(u) = \cos\left(\frac{2\pi ku}{n}\right)$

  for $k \leq n/2$. Both have eigenvalue $2 - 2\cos\left(\frac{2\pi k}{n}\right)$. Note $x_0$ should be ignored and $y_0$ is the all ones vector. If $n$ is even, then $x_{n/2}$ should be ignored.

Proof: By plotting the graph on the circle using these vectors as coordinates.
Let \( z(u) \) be the point \((x_k(u), y_k(u))\) on the plane.

Consider the vector \( z(u-1) - 2z(u) + z(u+1) \). By the reflection symmetry of the picture, it is parallel to \( z(u) \).

Let \( z(u-1) - 2z(u) + z(u+1) = \lambda z(u) \). By rotational symmetry, the constant \( \lambda \) is independent of \( u \).

To compute \( \lambda \) consider the vertex \( u=1 \).
The Path Graph

\[ P_n : \{(u, u+1) : 0 \leq u < n\} \]

- The Laplacian of \( P_n \) has the same eigenvalues as \( R_{2n} \) and eigenvectors \( z_k(u) = \sin\left(\frac{\pi ku}{n} + \frac{\pi}{2}\right) \), for \( k < n \).

   Proof: Treat \( P_n \) as a quotient of \( R_{2n} \). Use projection

\[
f : R_{2n} \to P_n
\]

\[
f(u) = \begin{cases} 
  u, & \text{if } u < n \\
  2n - 1 - u, & \text{if } u \geq n
\end{cases}
\]
The Path Graph

Proof: Treat $P_n$ as a quotient of $R_{2n}$. Use projection $f: R_{2n} \to P_n$

$$f(u) = \begin{cases} 
    u, & \text{if } u < n \\
    2n - 1 - u, & \text{if } u \geq n 
\end{cases}$$

- Let $z$ be an eigenvector of the ring, with $z(u) = z(2n-1-u)$ for all $u$.
- Take the first $n$ components of $z$ and call this vector $v$.
- To see that $v$ is an eigenvector of $P_n$, verify that it satisfies for some $\lambda$:
  - $2v(u) - v(u-1) - v(u+1) = \lambda v(u)$, for $0 < u < n-1$
  - $v(0) - v(1) = \lambda v(1)$
  - $v(n-1) - v(n-2) = \lambda v(n-1)$

- Take $z$ as claimed, i.e. $z_k(u) = \sin\left(\frac{\pi ku}{n} + \frac{\pi}{2n}\right)$, which is in the span of $x_k$ and $y_k$.

- (verify details as exercise)
Graph Products

- (Definition): Let $G(V,E)$ and $H(W,F)$. The graph product $G \times H$ is a graph with vertex set $V \times W$ and edge set $((v_1,w),(v_2,w))$ for $(v_1, v_2) \in E$
  $((v, w_1), (v, w_2))$ for $(w_1, w_2) \in F$
- If $G$ has evals $\lambda_1, \ldots, \lambda_n$, evects $x_1, \ldots, x_n$
  $H$ has evals $\mu_1, \ldots, \mu_m$, evects $y_1, \ldots, y_m$
  Then $G \times H$ has for all $i,j$ in range, an evector
  $z_{ij}(v,w) = x_i(v)y_j(w)$ of evalue $\lambda_i + \mu_j$
- Proof: see blackboard
Graph Products: Grid Graph

\[ G_{n \times m} = P_n \times P_m \]

- Immediately get spectra from path.
Start Bounding Laplacian Eigenvalues
Sum of Eigenvalues, Extremal Eigenvalues

- $\sum_i \lambda_i = \sum_i d_i \leq d_{\text{max}} n$ where $d_i$ is the degree of vertex $i$.
  
  Proof: take the trace of $L$

- $\lambda_2 \leq \frac{\sum_i d_i}{n-1}$ and $\lambda_n \geq \frac{\sum_i d_i}{n-1}$
  
  Proof: previous inequality + $\lambda_1 = 0$. 
Courant-Fischer

For any nxn symmetric matrix $A$ with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \ldots, v_n$, denote $S_k$ the span of $v_1, v_2, \ldots, v_k$ and $S_k^\perp$ the orthogonal complement, then

$$\alpha_k = \max_{x \in S_{k-1}^\perp \setminus \{0\}} \frac{x^T Ax}{x^T x}$$

$$\alpha_1 = \max_{x \neq 0} \frac{x^T Ax}{x^T x}$$

Proof: see blackboard
Courant-Fischer

- **Courant-Fischer Min Max Formula:** For any nxn symmetric matrix $A$ with eigenvalues $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ (decreasing order) and corresponding eigenvectors $v_1, v_2, \ldots, v_n$, denote $S_k$ the span of $v_1, v_2, \ldots, v_k$ and $S_k^\perp$ the orthogonal complement, then

$$\alpha_k = \max_{S \subseteq \mathbb{R}^n, \dim(S) = k} \min_{x \in S} \frac{x^T Ax}{x^T x}$$

$$\alpha_k = \min_{S \subseteq \mathbb{R}^n, \dim(S) = n-k+1} \max_{x \in S} \frac{x^T Ax}{x^T x}$$

Proof: see blackboard
Courant-Fischer for Laplacian

• Courant-Fischer Min Max Formula for increasing evale value order (e.g. Laplacians): For any nxn symmetric matrix L, with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) in increasing order

\[
\lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T L x}{x^T x}
\]

\[
\lambda_k = \max_{S \text{ of dim } n-k-1} \min_{x \in S} \frac{x^T L x}{x^T x}
\]

• Definition (Rayleigh Quotient): The ratio \( \frac{x^T L x}{x^T x} \) is called the Rayleigh Quotient of x with respect to L.

• Next lecture we will use it to bound evale values of Laplacians of certain graphs.