



CS 598: Spectral Graph Theory. Lecture 2

Extremal Eigenvalues and Eigenvectors of the Laplacian and the Adjacency Matrix.

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Today

- More on Courant-Fischer and Rayleigh quotients
- Applications of Courant-Fischer
- Adjacency matrix vs. Laplacian
- Perron-Frobenius

Courant-Fischer Refresher (1)

- **Courant-Fischer Min Max Formula:** For any $n \times n$ symmetric matrix A with eigenvalues $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ (decreasing order)

$$\alpha_k = \max_{S \subseteq \mathbb{R}^n, \dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_k = \min_{S \subseteq \mathbb{R}^n, \dim(S)=n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}$$

- Last time we saw proof, now we will see some applications
 - Sylvester's Law of Inertia
 - Bounds on Laplacian values

Sylvester's Law of Inertia

$$\alpha_k = \max_{S \subseteq \mathbb{R}^n, \dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

- **Theorem:** Let A be any symmetric matrix and B be any non-singular matrix. Then, the matrix BAB^T has the same number of positive, negative and zero eigenvalues as A .

Proof: see blackboard

Courant-Fischer Refresher (2)

- **Courant-Fischer Min Max Formula for increasing evalue order (e.g. Laplacians):** For any $n \times n$ symmetric matrix L , with eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ (in increasing order)

$$\alpha_k = \max_{S \subseteq R^n, \dim(S)=k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_k = \min_{S \subseteq R^n, \dim(S)=n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}$$

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$$\lambda_k = \min_{S \subseteq \mathbb{R}^n, \dim(S)=k} \max_{x \in S} \frac{x^T L x}{x^T x}$$

$$\alpha_k = \min_{S \subseteq \mathbb{R}^n, \dim(S)=n-k+1} \max_{x \in S} \frac{x^T A x}{x^T x}$$

Courant-Fischer Refresher (2)

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Courant-Fischer for Laplacian

- Applying Courant-Fischer for the Laplacian we get :

$$\lambda_k = \min_{S \text{ of dim } k} \max_{x \in S} \frac{x^T Lx}{x^T x}$$

$$\lambda_1 = 0, v_1 = 1$$

$$\lambda_2 = \min_{x \perp 1, x \neq 0} \frac{x^T Lx}{x^T x} = \min_{x \perp 1, x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

$$\lambda_{\max} = \max_{x \neq 0} \frac{x^T Lx}{x^T x} = \max_{x \neq 0} \frac{\sum_{(i,j) \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2}$$

- Useful for getting bounds, if calculating spectra is cumbersome.
- To get upper bound on λ_2 , just need to produce vector with small Rayleigh Quotient.
- Similarly, to get lower bound on λ_{\max} , just need to produce vector with large Rayleigh Quotient

Example 1

- *Lemma 1:* Let $G=(V,E)$ be a graph with some vertex w having degree d . Then

$$\lambda_{\max} \geq d$$

- *Lemma 2:* We can also improve on that. Under same assumptions, we can show:

$$\lambda_{\max} \geq d + 1$$

Proof: see blackboard

Example 1

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- *Lemma 2:* We can also improve on that. Under same assumptions, we can show:

$$\lambda_{\max} \geq d + 1$$

Lemma 2 is tight, take star graph (ex)

Example 2

- The Path graph P_n on n vertices has

$$\lambda_2 \leq \frac{12}{n^2}$$

- Already knew that, but this is easier and more general.

Proof: see blackboard

Example 3

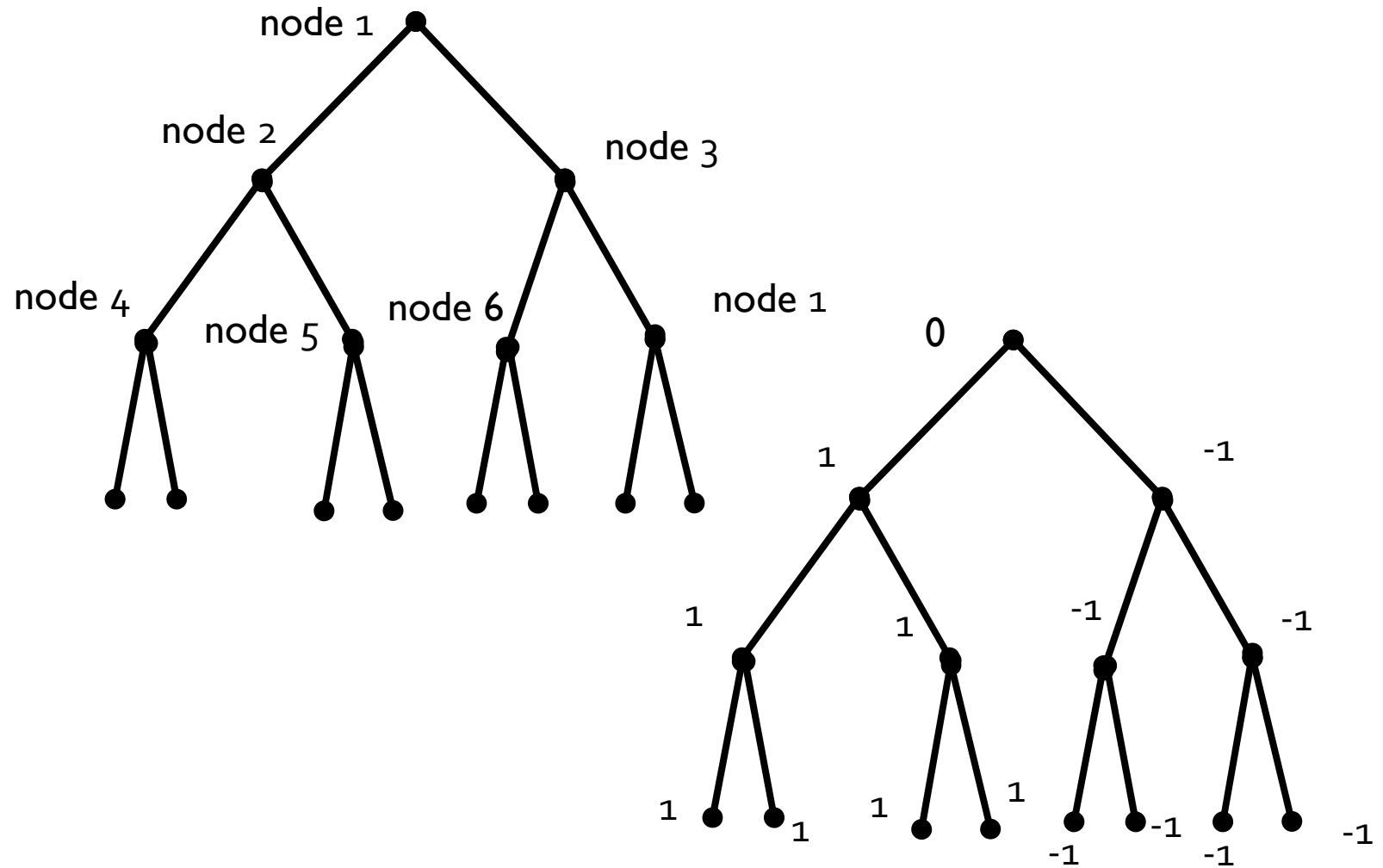
- The complete binary tree B_n on $n = 2^d - 1$ vertices has


$$\lambda_2 \leq \frac{2}{n-1}$$

B_n is the graph with edges of the form $(u, 2u)$ and $(u, 2u+1)$ for $u < n/2$.

Proof: See blackboard

Example 3

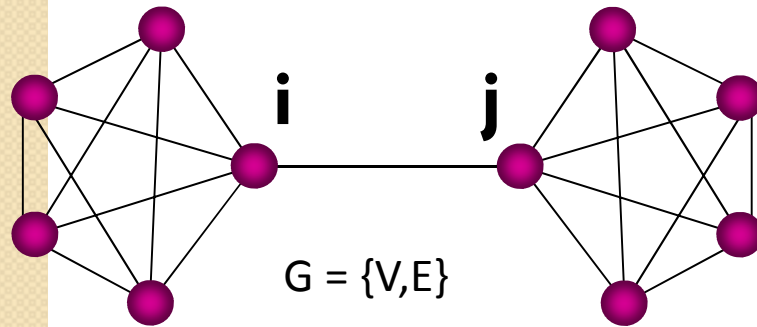


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- Lower bounds are harder, we will see some in two lectures (different technique)



Adjacency Matrix vs. Laplacian

Adjacency Matrix Refresher



- Unweighted graphs for simplicity

A has n eigenvalues (counting multiplicities)
 $\{a_1 \geq a_2 \geq \dots \geq a_n\}$

$$A_{ij} = \begin{cases} 1, & \text{if } (i, j) \text{ edge} \\ 0 & \text{if no edge between } i, j \end{cases}$$

		j	
		↓	
i		→ 1	

- Adjacency matrix as operator:

$$(A_G \mathbf{u})(i) = \sum_{j:(i,j) \in E} v(j)$$

Adjacency Matrix vs. Laplacian for d-regular graphs

- G is d-regular if every vertex has degree d. In this case: $L_G = D_G - A_G = dI - A_G$
- Let $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of L and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$ be the eigenvalues of A.
- We have $\alpha_i = d - \lambda_i$ and the corresponding eigenvectors are the same.

Bounds on the Eigenvalues of Adjacency Matrix

- $\alpha_1 \leq d_{\max}$

Proof: See blackboard.

- Adjacency matrix as operator:

$$(A_G \mathbf{u})(i) = \sum_{j:(i,j) \in E} \mathbf{u}(j)$$



Bounds on the Eigenvalues of Adjacency Matrix

- $\alpha_1 \leq d_{\max}$ with equality iff graph is d_{\max} – regular. In this case, the first eigenvector is the all-one's vector. (exercise)

Courant-Fischer for Adjacency Matrix Refresher

$$\alpha_k = \max_{S \text{ of dim } k} \min_{x \in S} \frac{x^T A x}{x^T x}$$

$$\alpha_1 = \max_{x \neq 0} \frac{x^T A x}{x^T x}$$

- Will see next how to apply Courant-Fischer for the adjacency matrix to get another bound on the first eigenvalue as well as a relation to graph coloring

Bounding Adjacency Matrix Eigenvalues

- **Lemma 1:** α_1 is at least the average degree of the vertices in G

Proof: see blackboard

$$\alpha_1 = \max_{x \neq 0} \frac{x^T Ax}{x^T x}$$



Bounding Adjacency Matrix Eigenvalues

- **Lemma 1:** α_1 is at least the average degree of the vertices in G
- While we may think of α_1 as being related to the average degree, it behaves differently. If we remove the vertex of smallest degree in a graph, the average degree can increase. However, α_1 only decreases when we remove a vertex.
- **Lemma 2:** Let A be a symmetric matrix, let B be the matrix obtained by removing the last row and column from A and let b_1 be the largest eigenvalue of B . Then $\alpha_1 \geq b_1$

Proof: see blackboard

Chromatic Number

- The chromatic number of a graph G , denoted $\chi(G)$, is the least k for which G has a k -coloring.
- **Theorem (Wilf):** $\chi(G) \leq \lfloor \alpha_1 \rfloor + 1$

Proof: see blackboard

Chromatic Number

- The chromatic number of a graph G , denoted $\chi(G)$, is the least k for which G has a k -coloring.
- **Theorem (Wilf):** $\chi(G) \leq \lfloor \alpha_1 \rfloor + 1$
- Improvement over classical bound $\chi(G) \leq d_{\max} + 1$, as there are graphs (e.g. path graph) where α_1 is much less than d_{\max}

Adjacency Matrix: The Perron-Frobenius Theorem

- We saw what happens for regular graphs. What if G is not regular? We know that $\alpha_1 < d_{\max}$ but what about v_1 ?

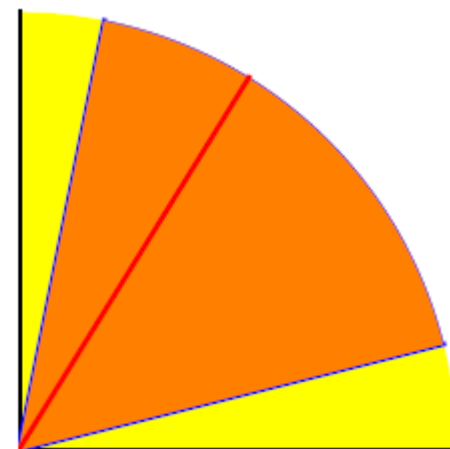
Perron-Frobenius Theorem (for graphs): Let $G=(V,E,w)$ be a connected graph, and let A be a non-negative matrix such that $A(i, j) > 0$ for all (i,j) edges. Then, there exists a positive vector v and a positive α such that $Av=\alpha v$. Moreover, the eigenvalue α is a unique maximal eigenvalue of A .

Adjacency Matrix: The Perron-Frobenius Theorem

- Proof in 3 parts.
- We next show

Lemma 1: If all entries of an $n \times n$ matrix A are positive, then it has a positive eigenvector v with corresponding positive eigenvalue α , that is $Av = \alpha v$.

Geometric proof on blackboard



Adjacency Matrix: The Perron-Frobenius Theorem

Lemma 1: If all entries of an $n \times n$ matrix A are positive, then it has a positive eigenvector v with corresponding positive eigenvalue α , that is $Av = \alpha v$.

From lemma 1, we derive the following (exercise)
(hint: consider powers of A)

Lemma 2: Let $G=(V,E,w)$ be a connected graph, and let A be a non-negative matrix such that $A(i,j) > 0$ for all (i,j) edges. Then, there exists a positive vector v and a positive α such that

$$Av = \alpha v.$$

Adjacency Matrix: The Perron-Frobenius Theorem

We conclude the Perron-Frobenius proof with the following lemma:

Lemma 3: Let $G=(V,E,w)$ be a connected graph. Assume that there is a positive vector v such that $Av = \alpha v$. Then

- (a) There is a non-negative, non-singular, diagonal matrix S such that $S^{-1}AS = \alpha I$
- (b) For every other eigenvalue α_i of A , $|\alpha_i| < \alpha$
- (c) The eigenvalue α has multiplicity 1.

Point (a) is example of matrix-scaling, useful for many applications

Laplacian: The Perron-Frobenius Theorem

- Theory can also be applied to Laplacians and any matrix with non-positive off-diagonal entries. It involves the eigenvector with smallest eigenvalue.

Perron-Frobenius for Laplacians: Let M be a matrix with non-positive off-diagonal entries s.t. the graph of the non-zero off-diagonal entries is connected. Then the smallest eigenvalue has multiplicity 1 and the corresponding eigenvector is strictly positive.

Next time we will see how to apply Perron Frobenius to show Fiedler's nodal domain theorem.