CS 598. Spectral Graph Theory
Problem Set 1

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Problem 1 *Adjacency Matrices*

(3 pts.)

1. Let $G = (V, E)$ be a d-regular, bipartite graph. Prove that $-d$ is an eigenvalue of A_G.
2. Let $G = (V, E)$ be a d-regular graph. Prove that if $-d$ is and eigenvalue of A_G, then G is bipartite.
3. Compute the eigenvalues and eigenvectors of the adjacency matrix of the star graph.

Problem 2 *Courant-Fischer*

(3 pts.)

1. Prove the Eigenvalue Interlacing Theorem for decreasing order of eigenvalues (Lecture 3). Namely, show that if A is an n-by-n symmetric matrix and B a principal submatrix of A of dimension n-1 (that is, B is obtained by deleting the same row and column from A). Then $\alpha_1 \geq \beta_1 \geq \cdots \geq \alpha_{n-1} \geq \beta_{n-1}$ Where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-1}$ are the eigenvalues of A and B respectively.
2. Show that if A is an n-by-n symmetric matrix and B be a principal submatrix of A of dimension n-k (that is, B is obtained by deleting the same set of k rows and columns from A). Then $\alpha_i \geq \beta_i \geq \alpha_{i+k}$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_{n-k}$ are the eigenvalues of A and B respectively.
3. With the same assumptions as in item (2), show that if instead, we order the eigenvalues of A and B in increasing order, that is $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_{n-k}$ then $\alpha_i \leq \beta_i \leq \alpha_{i+k}$.

Problem 3 *Perron-Frobenius*

(3 pts.)

We recall Lemma 1 and Lemma 2 from slide 27, in the class slides for Lecture 8. We re-cast the theorems in the following equivalent form, in order to get familiar with the matrix scaling technique:

Lemma 1. *Let A be a matrix such that $A(i, j) > 0$ for all i and j. Then there exists a positive vector $\mathbf{v}$ and an $\alpha > 0$ such that*

$$S^{-1}AS\mathbf{1} = \alpha\mathbf{1}$$

where $S = diag(\mathbf{v})$.

Lemma 2. *Let $G = (V, E)$ be a connected graph and let A be a non-negative matrix such that $A(i, j) > 0$ for all $(i, j) \in E$. Then there exists a positive vector $\mathbf{v}$ and an $\alpha > 0$ such that*

$$S^{-1}AS\mathbf{1} = \alpha\mathbf{1}$$

where $S = diag(\mathbf{v})$.

Prove that Lemma 2 above follows from Lemma 1.

Problem 4 *Upper bounds on Eigenvalues and Graphic Inequalitites*

(6 pts.)

1. For d a positive integer and $n = 2^d$, let G_n be the graph with vertex set $\{0, 1\}^d$ in which every pair of vertices that differ in at most two coordinates are joined by an edge. Prove upper and lower bounds on $\lambda_2(G_n)$. Make them as close to each other as possible.

2. For the complete binary tree T_n, prove that $\lambda_2(T_n) \geq 1/cn$ for some absolute constant c.

3. Let $w_1, \cdots, w_{n-1} > 0$ and let P be a weighted path with Laplacian $L_P = \sum_{i=1}^{n-1} w_i L_{(i,i+1)}$. Let v be any vector such that

$$v(1) < v(2) < \cdots < v(n)$$

Prove that

$$\lambda_2(P) \geq min_{i:v_i \neq 0} \frac{(L_P v)_i}{v_i}$$

This shows that a test vector can actually be used to prove a lower bound on λ_2!!