Spring 2022, CS 586/IE 519: Combinatorial Optimization
Homework 4
Due: Thursday, April 14, 2022

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince us that you know the solution, as quickly as possible.

Submit solutions to at least four problems.

Problem 1 Given a graph $G = (V, E)$ let $\mathcal{I} = \{S \subseteq V \mid \text{there is a matching } M \text{ in } G \text{ that covers } S\}$. Prove that $(V, \mathcal{I})$ is a matroid.

Problem 2 Let $G = (V, E)$ be a graph and let $\mathcal{I} = \{J \subseteq E : |J \cap E(U)| \leq 2|U| - 3, U \subseteq V, |U| > 1\}$. Show that $(E, \mathcal{I})$ is a matroid.

Problem 3 Let A be a full-rank $n \times n$ matrix over the reals. Let R and C be the index sets of the rows and columns. Given $I \subset R$ show that there exists $J \subset C$ such that $|I| = |J|$ and both $A(I, J)$ and $A(R \setminus I, C \setminus J)$ are of full rank. Use matroid intersection.

Problem 4 We saw that matroid union can be algorithmically reduced to matroid intersection. One can derive the matroid intersection theorem from matroid union theorem. Givem two matroids $\mathcal{M}_1 = (S, \mathcal{I}_1)$ and $\mathcal{M}_2 = (S, \mathcal{I}_2)$ let $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2^*$ be the union of $\mathcal{M}_1$ and the dual of $\mathcal{M}_2$. Let B be a base of $\mathcal{M}$; it follows that B can be partitioned into J_1 and J_2 where J_1 is independent in $\mathcal{M}_1$ and J_2 is independent in $\mathcal{M}_2^*$. Extend J_2 (in an arbitrary way) to a base B_2 in $\mathcal{M}_2^*$. Show that $B \setminus B_2$ is maximum cardinality common independent set of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Problem 5 Consider the following graph orientation problem: given an undirected graph $G = (V, E)$ and a positive integer k, determine if there exists an orientation of the edges of G such that each vertex has in-degree at most k.

- Formulate this problem as a matroid intersection problem.
- Use the min-max relation for the max cardinality of a common independent set of two matroids to show that such an orientation exists if and only if $|E[S]| \leq k|S|$ for every subset $S \subseteq V$.