Spring 2022, CS 586/IE 519: Combinatorial Optimization
Homework 2
Due: Tuesday, Feb 22nd, 2022

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince us that you know the solution, as quickly as possible.

Solve Problems 1,6 and two other problems.

Problem 1 Consider the LP $\max c^T x$ such that $Ax \leq b, x \in \mathbb{R}^n$ where A is a rational $m \times n$ matrix and c is $n \times 1$ rational vector and b is a rational $m \times 1$ vector. Prove that if the optimum value is finite for the LP then it is achieved by a vector y that is contained in a minimal face of the polyhedron $P = \{x \mid Ax \leq b\}$. This problem is meant to be easy and it mainly to make you read some relevant definitions.

Problem 2 Let $G = (V, E)$ be a bipartite graph and let k be an integer. Let S_k be the set of characterstic vectors of the matchings in G of size at most k. Show that the following polytope is the convex hull of the vectors in S_k. Alternatively, show that it is an integral polytope.

$$\begin{aligned} \sum_{e \in E} x(e) &\leq k \\ x(\delta(u)) &\leq 1 \qquad u \in V \\ x(e) &\geq 0 \qquad e \in E \end{aligned}$$

Problem 3 Consider a bipartite graph $G = (V, E)$ with A, B as the bipartition of the vertex set B. A subset $X \subseteq A$ is matchable if there is matching that saturates X. Suppose X, Y are matchable and $|X| < |Y|$. Show that there is a $y \in Y \setminus X$ such that $X' = X \cup \{y\}$ is also matchable.

Problem 4 Consider a bipartite graph $G = (V, E)$ with A, B as the bipartition of the vertex set B. For $X \subseteq A$, define $def(X)$ as $|X| - |N(X)|$ where $N(X)$ is the set of neighbors of X in B. Generalize Hall's theorem to show that the size of a maximum matching in G is equal to $|A| - k$ where $k = \max_{X \subseteq A} def(X)$. Note that $k \geq 0$ by taking $X = \emptyset$.

Problem 5 Let M be a matrix. Show that the maximum number of non-zero entries in M with no two in the same line (i.e., same row or same column) is equal to the minimum number of lines that contain all nonzero entries.

Problem 6 Let $T = (V, E)$ be a rooted tree with non-negative edge costs $c : E \to \mathbb{Z}_+$. Let $(s_1, t_1), \ldots, (s_k, t_k)$ be a given set of vertex pairs from $V \times V$. A set of edges $A \subseteq E$ in the tree is a *multicut* for the given pairs if in the forest $T - A$ there is no path from s_i to t_i for any $i \in [k]$. The multicut problem is NP-Hard even in trees. However we can solve a special case and that leads to a 2-approximation in trees. We will focus on the special case.

- Write an integer programming formulation for the multicut problem in trees with binary variables $x_e, e \in E$.
- Reduce the minimum vertex cover problem in a general graph to the multicut problem in a star. Use this insight to argue that the LP relaxation is not integral even in stars.
- Suppose we have a special case of multicut in a tree where s_i is an ancestor of t_i in T or the other way around (here we fix a specific rooting). Prove that the underlying matrix in the LP relaxation is a network matrix and hence the LP is integral.
- **Extra credit:** Use the preceding part to argue that the LP relaxation in general trees can be used to obtain a 2-approximation. *Hint:* Solve the LP and use it to separate s_i or t_i from their least common ancestor by losing a factor of 2 in the LP cost and apply preceding part.